1.1 Equations and Graphs

Learning Objectives

A student will be able to:

- Find solutions of graphs of equations.
- Find key properties of graphs of equations including intercepts and symmetry.
- Find points of intersections of two equations.
- Interpret graphs as models.

Introduction

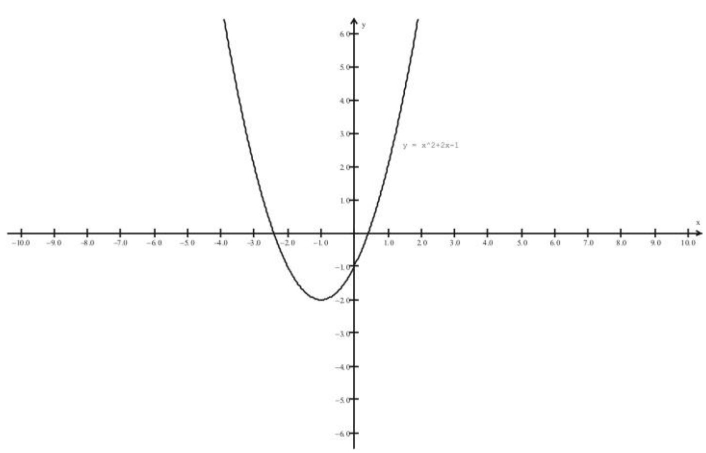
In this lesson we will review what you have learned in previous classes about mathematical equations of relationships and corresponding graphical representations and how these enable us to address a range of mathematical applications. We will review key properties of mathematical relationships that will allow us to solve a variety of problems. We will examine examples of how equations and graphs can be used to model real-life situations.

Let's begin our discussion with some examples of algebraic equations:

Example 1: $y = x^2$. The equation has ordered pairs of numbers (x, y) as solutions. Recall that a particular pair of numbers is a solution if direct substitution of the x and y values into the original equation yields a true equation statement. In this example, several solutions can be seen in the following table:

x	$y = x^2 + 2x - 1$
-4	7
-3	2
-2	-1
-1	-2
0	-1
1	2
2	7
3	14

We can graphically represent the relationships in a rectangular coordinate system, taking the x as the horizontal axis and the y as the vertical axis. Once we plot the individual solutions, we can draw the curve through the points to get a sketch of the graph of the relationship:



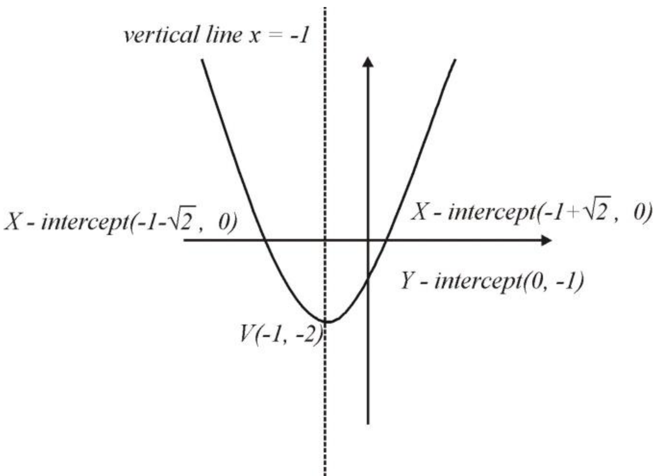
We call this shape a parabola and every quadratic function, $f(x) = ax^2 + bx + c, a \neq 0$ has a parabola-shaped graph. Let's recall how we analytically find the key points on the parabola. The vertex will be the lowest point, $(-1, -2)$. In general, the vertex is located at the point $(b/2a, f(b/2a))$. We then can identify points crossing the $x$and $y$axes. These are called the intercepts of the equation. The $y-$intercept is found by setting $x = 0$in the equation, and then solving for $y$as follows:

$y = 0^2 + 2(0) - 1 = -1$. The $y-$intercept is located at $(0, -1)$.

The $x-$intercept is found by setting $y = 0$in the equation, and solving for $x$as follows: $0 = x^2 + 2x - 1$

Using the quadratic formula, we find that $x = -1 \pm \sqrt{2}$. The $x-$intercepts are located at $(-1 - \sqrt{2}, 0)$and $(-1 + \sqrt{2}, 0)$.

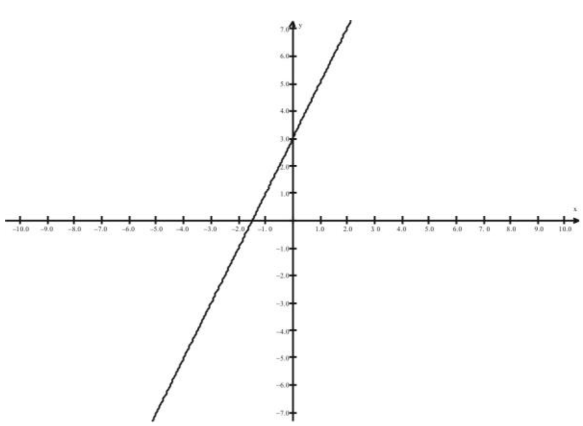
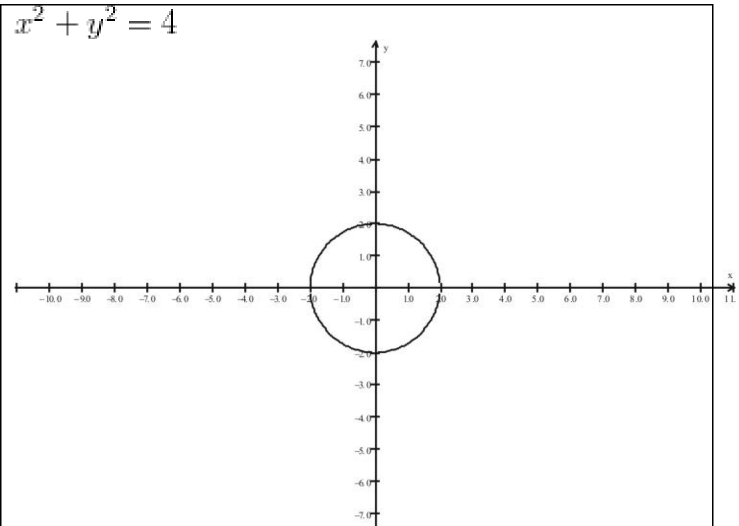
Finally, recall that we defined the symmetry of a graph. We noted examples of vertical and horizontal line symmetry as well as symmetry about particular points. For the current example, we note that the graph has symmetry in the vertical line $x = -1$.The graph with all of its key characteristics is summarized below:

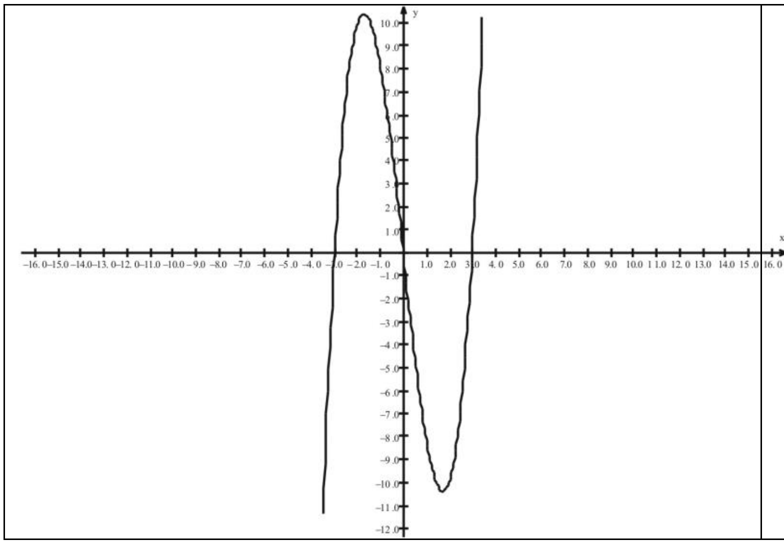


Let’s look at a couple of more examples.

Example 2:

Here are some other examples of equations with their corresponding graphs:

$y = 2x + 3$ 	$x^2 + y^2 = 4$ 
$y = x^3 - 9x$	



Example 3:

We recall the first equation as linear so that its graph is a straight line. Can you determine the intercepts?

Solution:

$x-$intercept at $(-3/2, 0)$ and $y-$intercept at $(0, 3)$.

Example 4:

We recall from pre-calculus that the second equation is that of a circle with center $(0, 0)$ and radius $r = 2$. Can you show analytically that the radius is 2?

Solution:

Find the four intercepts, by setting $x = 0$ and solving for y, and then setting $y = 0$ and solving for x.

Example 5:

The third equation is an example of a polynomial relationship. Can you find the intercepts analytically?

Solution:

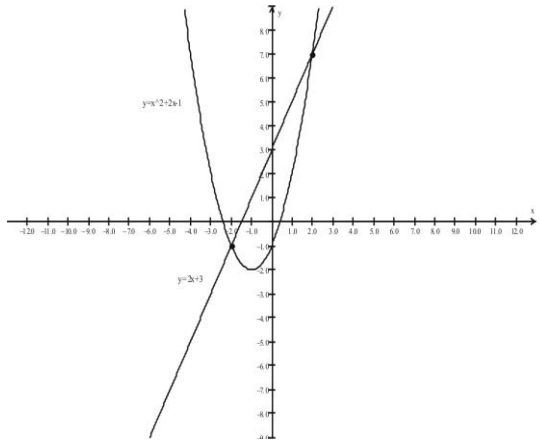
We can find the $x-$intercepts analytically by setting $y = 0$ and solving for x. So, we have

$$\begin{aligned} x^3 - 9x &= 0 \\ x(x^2 - 9) &= 0 \\ x(x-3)(x+3) &= 0 \\ x = 0, x = -3, x = 3. \end{aligned}$$

So the $x-$intercepts are located at $(-3, 0)$, $(0, 0)$, and $(3, 0)$. Note that $(0, 0)$ is also the $y-$intercept. The $y-$intercepts can be found by setting $x = 0$. So, we have

$$
\begin{aligned}
x^3 - 9x &= y \\
(0)^3 - 9(0) &= y \\
y &= 0.
\end{aligned}
$$

Sometimes we wish to look at pairs of equations and examine where they have common solutions. Consider the linear and quadratic graphs of the previous examples. We can sketch them on the same axes:



We can see that the graphs intersect at two points. It turns out that we can solve the problem of finding the points of intersections analytically and also by using our graphing calculator. Let's review each method.

Analytical Solution

Since the points of intersection are on each graph, we can use substitution, setting the general $y-$coordinates equal to each other, and solving for x.

$$
\begin{aligned}
2x + 3 &= x^2 + 2x - 1 \\
0 &= x^2 - 4 \\
x = 2, x &= -2.
\end{aligned}
$$

We substitute each value of $x$into one of the original equations and find the points of intersections at $(-2, -1)$and $(2, 7)$.

Graphing Calculator Solution

Once we have entered the relationships on the **Y=** menu, we press **2nd [CALC]** and choose #5 **Intersection** from the menu. We then are prompted with a cursor by the calculator to indicate which two graphs we want to work with. Respond to the next prompt by pressing the left or right arrows to move the cursor near one of the points of intersection and press **[ENTER]**. Repeat these steps to find the location of the second point.

We can use equations and graphs to model real-life situations. Consider the following problem.

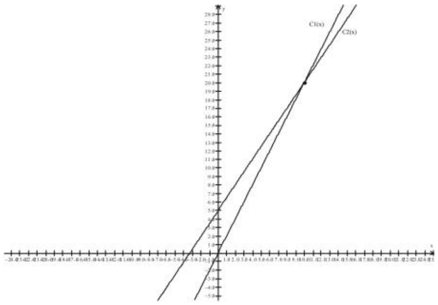
Example 6: Linear Modeling

The cost to ride the commuter train in Chicago is $\$2$. Commuters have the option of buying a monthly coupon book costing $\$5$that allows them to ride the train for $\$1.5$ on each trip. Is this a good deal for someone who commutes every day to and from work on the train?

Solution:

We can represent the cost of the two situations, using the linear equations and the graphs as follows:

$$C_1(x) = 2x$$
$$C_2(x) = 1.5x + 5$$



As before, we can find the point of intersection of the lines, or in this case, the break-even value in terms of days, by solving the equation:

$$\begin{aligned} C_1(x) &= C_2(x) \\ 2x &= 1.5x + 5 \\ x &= 10. \end{aligned}$$

So, even though it costs more to begin with, after 10days the cost of the coupon book pays off and from that point on, the cost is less than for those riders who did not purchase the coupon book.

Example 7: Non-Linear Modeling

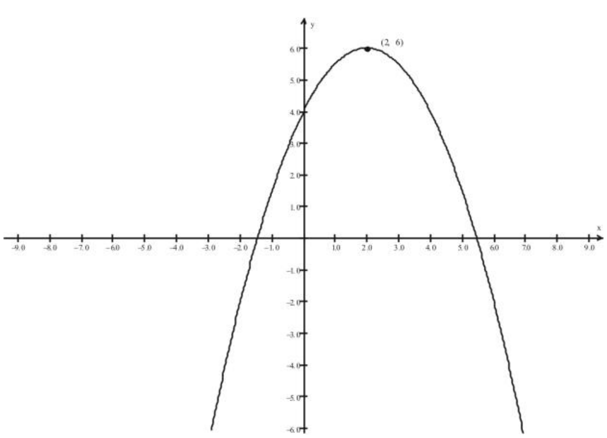
The cost of disability benefits in the Social Security program for the years 2000 - 2005 can be modeled as a quadratic function. The formula

$$Y = -0.5x^2 + 2x + 4$$

indicates the number of people Y, in millions, receiving Disability Benefits $x$years after 2000. In what year did the greatest number of people receive benefits? How many people received benefits in that year?

Solution:

We can represent the graph of the relationship using our graphing calculator.



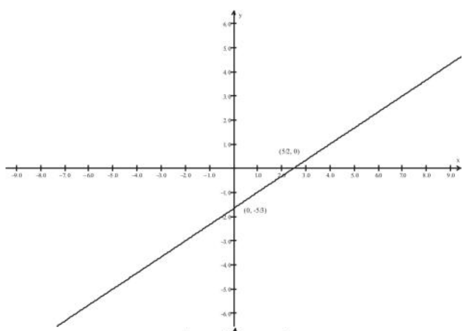
The vertex is the maximum point on the graph and is located at $(2, 6)$. Hence in year 2002 a total of **6 million** people received benefits.

Lesson Summary

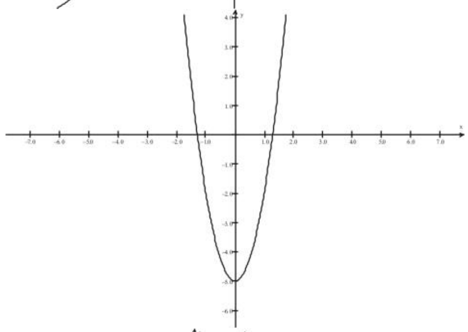
1. Reviewed graphs of equations
2. Reviewed how to find the intercepts of a graph of an equation and to find symmetry in the graph
3. Reviewed how relationships can be used as models of real-life phenomena
4. Reviewed how to solve problems that involve graphs and relationships

Review Questions

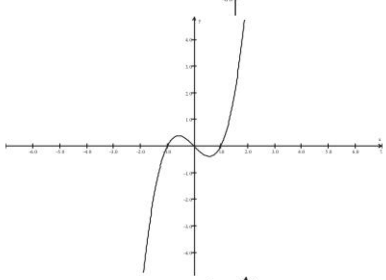
In each of problems 1 - 4, find a pair of solutions of the equation, the intercepts of the graph, and determine if the graph has symmetry.



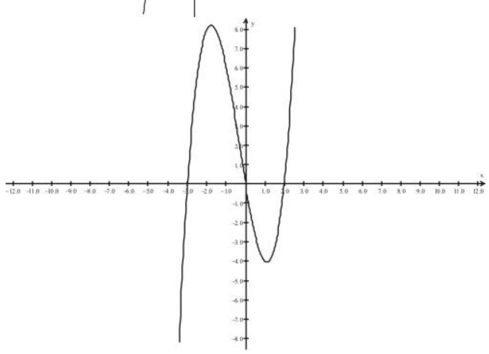
1. $2x - 3y = 5$



2. $3x^2 - y = 5$

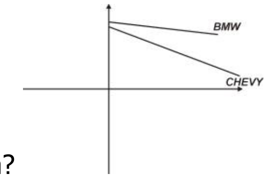


3. $y = x^3 - x$

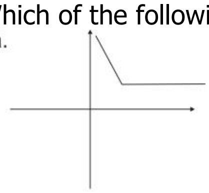


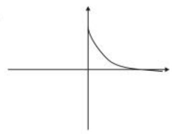
4. $y = x^3 + x^2 - 6x$

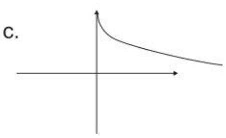
5. Once a car is driven off of the dealership lot, it loses a significant amount of its resale value. The graph below shows the depreciated value of a BMW versus that of a Chevy after t years. Which of the following statements is



the best conclusion about the data?
    a. You should buy a BMW because they are better cars.
    b. BMWs appear to retain their value better than Chevys.
    c. The value of each car will eventually be $0.
6. Which of the following graphs is a more realistic representation of the depreciation of cars.

a.


b.


c.


7. A rectangular swimming pool has length that is 25 yards greater than its width.
    a. Give the area enclosed by the pool as a function of its width.
    b. Find the dimensions of the pool if it encloses an area of 264 square yards.
8. Suppose you purchased a car in 2004 for $18,000. You have just found out that the current year 2008 value of your car is $8,500. Assuming that the rate of depreciation of the car is constant, find a formula that shows changing value of the car from 2004 to 2008.
9. For problem #8, in what year will the value of the vehicle be less than $1,400?
10. For problem #8, explain why using a constant rate of change for depreciation may not be the best way to model depreciation.

Review Answers

1. $(1,-1)$ and $(4,1)$ are two solutions. The intercepts are located at (0, -5/3) and (5/2, 0). We have a linear relationship between x and y, so its graph can be sketched as the line passing through any two solutions.
2. By solving for y, we have $y=3x^2-5,$ so two solutions are $(-1,-2)$ and $(1,-2)$. The x-intercepts are located at $\left(\pm\sqrt{\frac{5}{3}},0\right)$ and the y-intercept is located at $(0,-5)$. The graph is symmetric in the y-axis.
3. Using your graphing calculator, enter the relationship on the **Y=** menu. Viewing a table of points, we see many solutions, say $(2,6)$ and $(-2,-6),$ and the intercepts at $(0,0), (-1,0)$ and $(1,0)$. By inspection we see that the graph is symmetric about the origin.
4. Using your graphing calculator, enter the relationship on the **Y=** menu. Viewing a table of points, we see many solutions, say $(2,0),$ and $(-1,6),$ and the intercepts located at $(0,0), (-3,0),$ and $(2,0)$. By inspection we see that the graph does not have any symmetry.
5. b.
6. c. because you would expect (1) a decline as soon as you bought the car, and (2) the value to be declining more gradually after the initial drop.
7.
    a. $A(w)=w^2+25w.$
    b. The pool has area 264 when width = 8 and length = 33.
8. The rate of change will be $(-9500/4)=-2375.$ The formula will be $y=-2375x+18000.$
9. At the time x=7, or equivalently in the year 2011, the car will be valued at $1375.
10. A linear model may not be the best function to model depreciation because the graph of the function decreases as time increases; hence at some point the value will take on negative real number values, an impossible situation for the value of real goods and products.

1.2 Relations and Functions

Learning Objectives

A student will be able to:

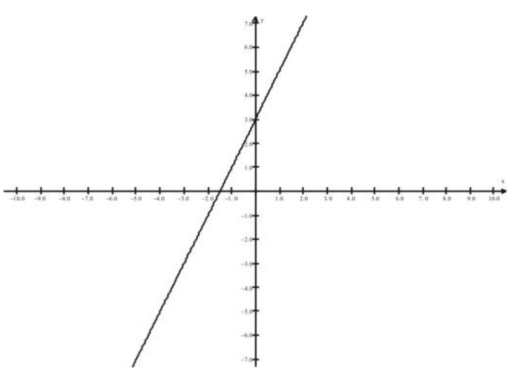
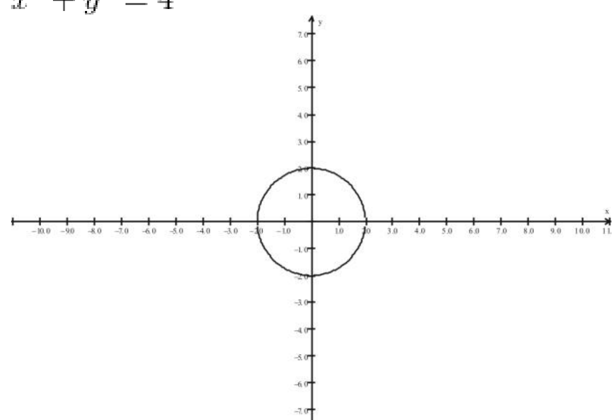
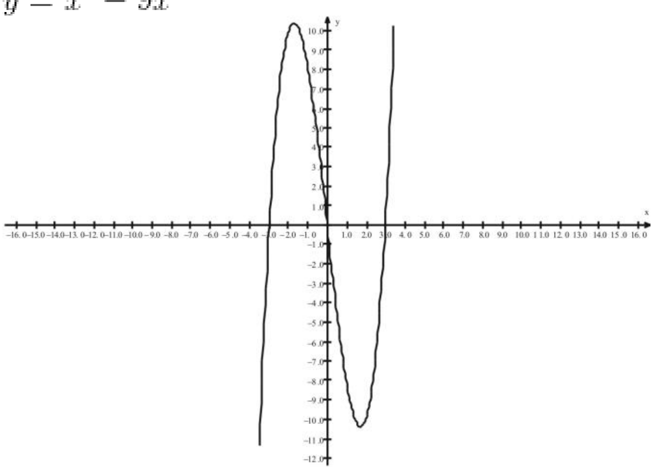
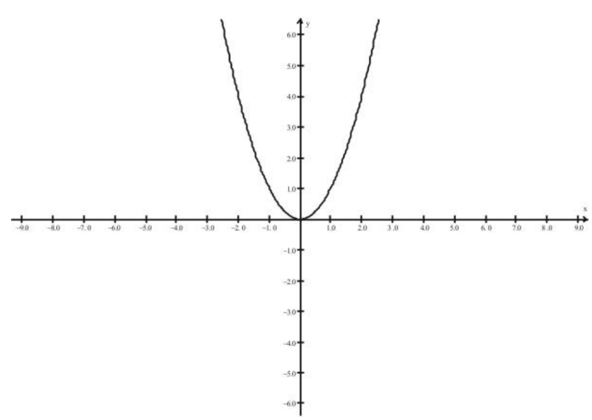
- Identify functions from various relationships.
- Review function notation.
- Determine domains and ranges of particular functions.
- Identify key properties of some basic functions.
- Sketch graphs of basic functions.
- Sketch variations of basic functions using transformations.
- Compose functions.

Introduction

In our last lesson we examined a variety of mathematical equations that expressed mathematical relationships. In this lesson we will focus on a particular class of relationships called functions, and examine their key properties. We will then review how to sketch graphs of some basic functions that we will revisit later in this class. Finally, we will examine a way to combine functions that will be important as we develop the key concepts of calculus.

Let's begin our discussion by reviewing four types of equations we examined in our last lesson.

Example 1:

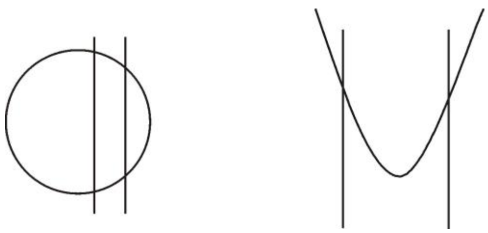
$y = 2x + 3$ 	$x^2 + y^2 = 4$ 
$y = x^3 - 9x$ 	$y = x^2$ 

Of these, the circle has a quality that the other graphs do not share. Do you know what it is?

Solution:

The circle's graph includes points where a particular $x-$value has two points associated with it; for example, the points $(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are both solutions to the equation $x^2 + y^2 = 4$. For each of the other relationships, a particular $x-$value has exactly one $y-$value associated with it.)

The relationships that satisfy the condition that for each $x-$value there is a unique $y-$value are called **functions**. Note that we could have determined whether the relationship satisfied this condition by a graphical test, the vertical line test. Recall the relationships of the circle, which is not a function. Let's compare it with the parabola, which is a function.



If we draw vertical lines through the graphs as indicated, we see that the condition of a particular $x-$value having exactly one $y-$value associated with it is equivalent to having at most one point of intersection with any vertical line. The lines on the circle intersect the graph in more than one point, while the lines drawn on the parabola intersect the graph in exactly one point. So this vertical line test is a quick and easy way to check whether or not a graph describes a function.

We want to examine properties of functions such as function notation, their domain and range (the sets of x and y values that define the function), graph sketching techniques, how we can combine functions to get new functions, and also survey some of the basic functions that we will deal with throughout the rest of this book.

Let's start with the notation we use to describe functions. Consider the example of the linear function $y = 2x + 3$. We could also describe the function using the symbol $f(x)$ and read as "f *of* x" to indicate the $y-$value of the function for a particular $x-$value. In particular, for this function we would write $f(x) = 2x + 3$ and indicate the value of the function at a particular value, say $x = 4$ as $f(4)$ and find its value as follows: $f(4) = 2(4) + 3 = 11$. This statement corresponds to the solution $(4, 11)$ as a point on the graph of the function. It is read, "f *of* x *is* 11."

We can now begin to discuss the properties of functions, starting with the ***domain*** and the ***range*** of a function. The ***domain*** refers to the set of $x-$values that are inputs in the function, while the ***range*** refers to the set of $y-$values that the function takes on. Recall our examples of functions:

Linear Function $g(x) = 2x + 3$

Quadratic Function $f(x) = x^2$

Polynomial Function $p(x) = x^3 - 9x$

We first note that we could insert any real number for an $x-$value and a well-defined $y-$value would come out. Hence each function has the set of all real numbers as a domain and we indicate this in interval form as $D : (-\infty, \infty)$. Likewise we see that our graphs could extend up in a positive direction and down in a negative direction without end in either direction. Hence we see that the set of $y-$values, or the range, is the set of all real numbers $R : (-\infty, \infty)$.

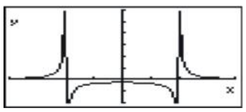
Example 2:

Determine the domain and range of the function.

$f(x) = 1/(x^2 - 4).$

Solution:

We note that the condition for each $y-$value is a fraction that includes an x term in the denominator. In deciding what set of $x-$values we can use, we need to exclude those values that make the denominator equal to 0.Why? **(Answer: division by 0 is not defined for real numbers.)** Hence the set of all permissible $x-$values, is all real numbers except for the numbers $(2, -2)$,which yield division by zero. So on our graph we will not see any points that correspond to these $x-$values. It is more difficult to find the range, so let's find it by using the graphing calculator to produce the graph.



From the graph, we see that every $y \neq 0$value in $(-\infty, \infty)$(or "All real numbers") is represented; hence the range of the function is $\{-\infty, 0\} \cup \{0, \infty\}$.This is because a fraction with a non-zero numerator never equals zero.

Eight Basic Functions

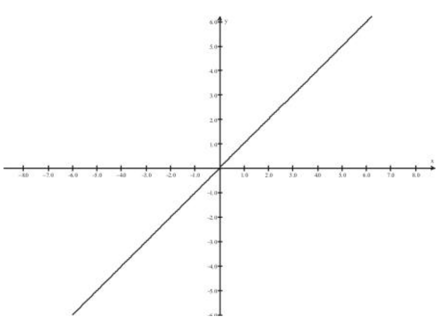
We now present some basic functions that we will work with throughout the course. We will provide a list of eight basic functions with their graphs and domains and ranges. We will then show some techniques that you can use to graph variations of these functions.

Linear

$f(x) = x$

Domain** $=$**All reals

Range** $=$**All reals

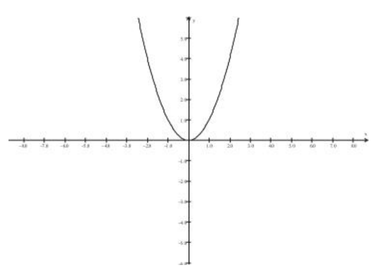


Square (Quadratic)

$f(x) = x^2$

Domain** $=$**All reals

Range $= \{y \geq 0\}$

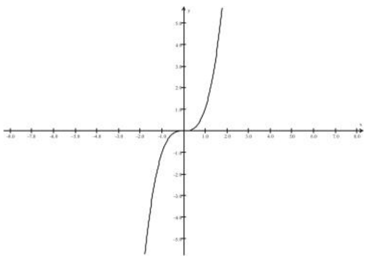


Cube (Polynomial)

$f(x) = x^3$

Domain $=$ ***All reals***

Range $=$ ***All reals***

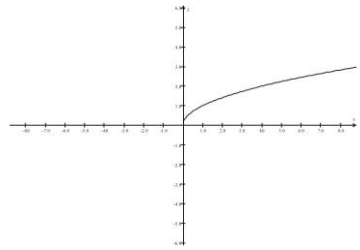


Square Root

$f(x) = \sqrt{x}$

Domain $= \{x \geq 0\}$

Range $= \{y \geq 0\}$

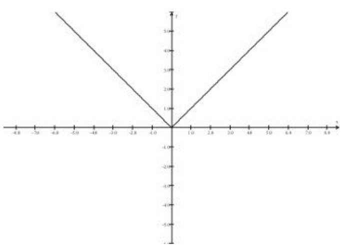


Absolute Value

$f(x) = |x|$

Domain $=$ ***All reals***

Range $= \{y \geq 0\}$

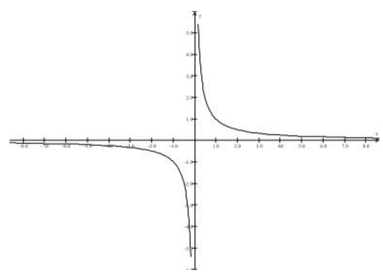


Rational

$f(x) = 1/x$

Domain $= \{x \neq 0\}$

Range $= \{y \neq 0\}$

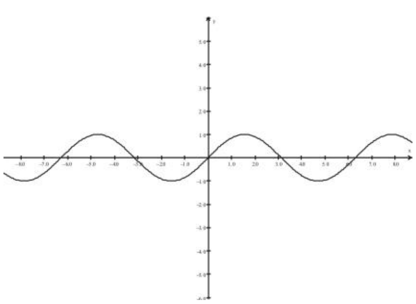


Sine

$f(x) = \sin x$

Domain $=$ ***All reals***

Range $= \{-1 \leq y \leq 1\}$

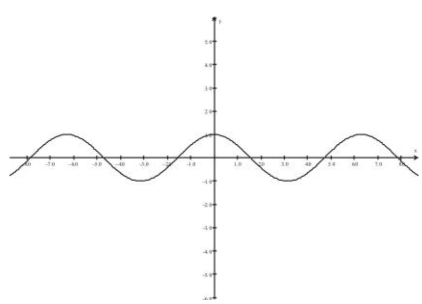


Cosine

$f(x) = \cos x$

Domain $=$ ***All reals***

Range $= \{-1 \leq y \leq 1\}$

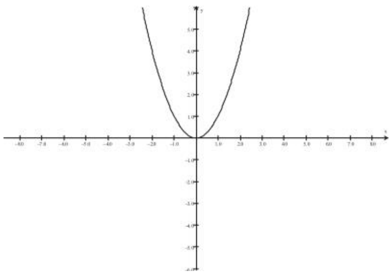


Graphing by Transformations

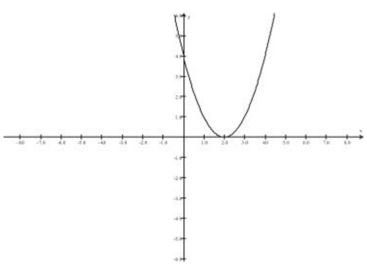
Once we have the basic functions and each graph in our memory, we can easily sketch variations of these. In general, if we have $f(x)$ and c is some constant value, then the graph of $f(x-c)$ is just the graph of $f(x)$ shifted c units to the right. Similarly, the graph of $f(x+c)$ is just the graph of $f(x)$ shifted c units to the left.

Example 3:

$f(x) = x^2$



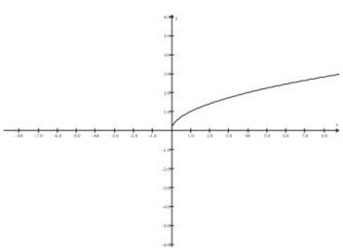
$f(x) = (x-2)^2$



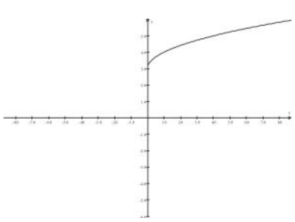
In addition, we can shift graphs up and down. In general, if we have $f(x)$ and c is some constant value, then the graph of $f(x)+c$ is just the graph of $f(x)$ shifted c units up on the $y-$axis. Similarly, the graph of $f(x)-c$ is just the graph of $f(x)$ shifted c units down on the $y-$axis.

Example 4:

$f(x) = \sqrt{x}$

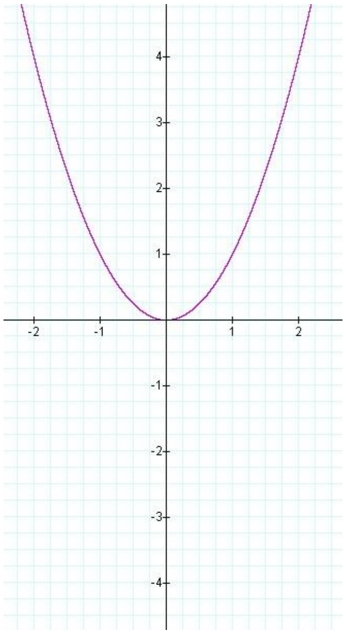


$f(x) = \sqrt{x} + 3$

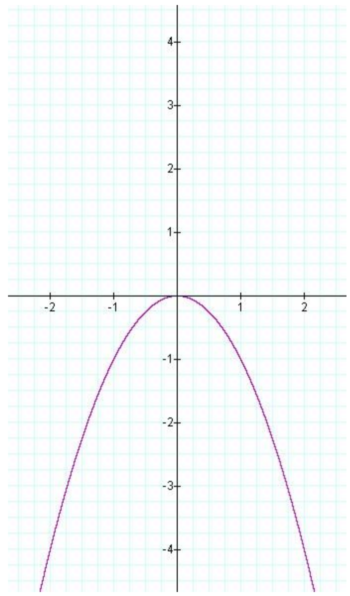


We can also flip graphs in the $x-$axis by multiplying by a negative coefficient.

$f(x) = x^2$



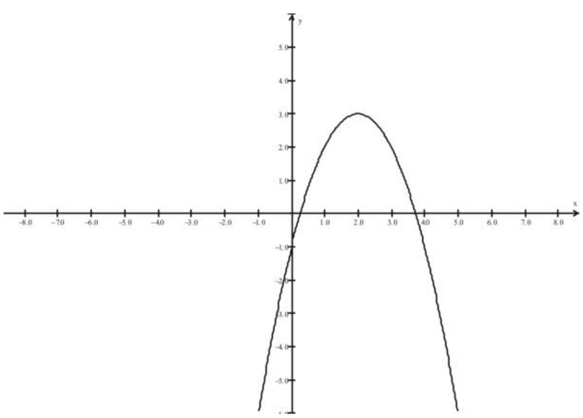
$f(x) = -x^2$



Finally, we can combine these transformations into a single example as follows.

Example 5:

$f(x) = -(x-2)^2 + 3$. The graph will be generated by taking $f(x) = x^2$, flipping in the $y-$axis, and moving it two units to the right and up three units.



Function Composition

The last topic for this lesson involves a way to combine functions called **function composition**. Composition of functions enables us to consider the effects of one function followed by another. Our last example of graphing by transformations provides a nice illustration. We can think of the final graph as the effect of taking the following steps:

$$x \to -(x-2)^2 \to -(x-2)^2 + 3$$

We can think of it as the application of two functions. First, $g(x)$ takes x to $-(x-2)^2$ and then we apply a second function, $f(x)$ to those $y-$values, with the second function adding $+3$ to each output. We would write the functions as

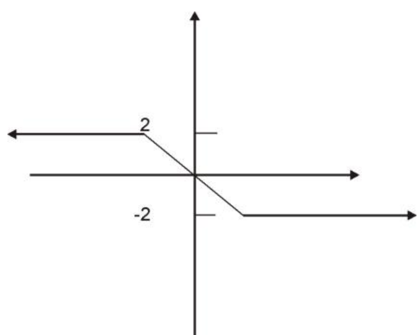
$f(g(x)) = -(x-2)^2 + 3$ where $g(x) = -(x-2)^2$ and $f(x) = x+3$. We call this operation the composing of f with g and use notation $f \circ g$. Note that in this example, $f \circ g \neq g \circ f$. Verify this fact by computing $g \circ f$ right now. (Note: this fact can be verified algebraically, by showing that the expressions $f \circ g$ and $g \circ f$ differ, or by showing that the different function decompositions are not equal for a specific value.)

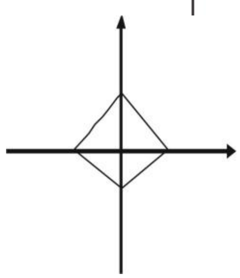
Lesson Summary

1. Learned to identify functions from various relationships.
2. Reviewed the use of function notation.
3. Determined domains and ranges of particular functions.
4. Identified key properties of basic functions.
5. Sketched graphs of basic functions.
6. Sketched variations of basic functions using transformations.
7. Learned to compose functions.

Review Questions

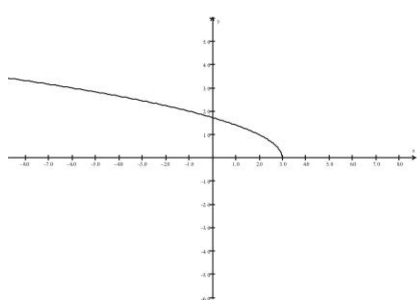
In problems 1 - 2, determine if the relationship is a function. If it is a function, give the domain and range of the function.

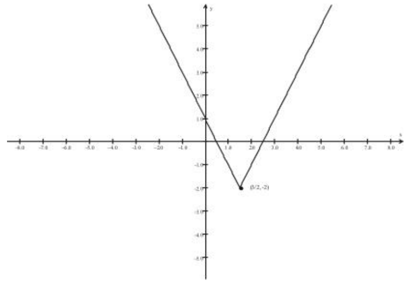
1. 

2. 

In problems 3 - 5, determine the domain and range of the function and sketch the graph if no graph is provided.

3. $f(x) = \frac{3x^2}{x^2-1}$ (no graph provided)

4. $y = \sqrt{-x+3}$ graph: 

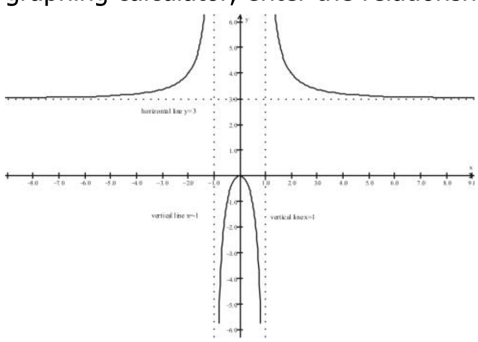
5. $f(x) = |2x - 3| - 2$ graph: 

In problems 6 - 8, sketch the graph using transformations of the graphs of basic functions.

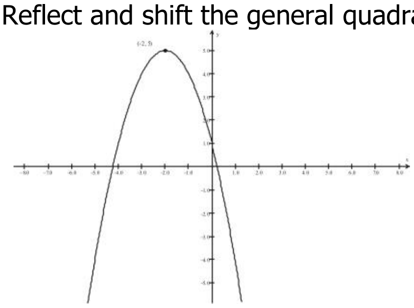
6. $f(x) = -(x+2)^2 + 5$
7. $f(x) = -\frac{1}{x-2} + 3$
8. $y = -\sqrt{-x-2} + 3$
9. Find the composites $f \circ g$ and $g \circ f$ for the following functions. $f(x) = -3x + 2,\ g(x) = \sqrt{x}$
10. Find the composites $f \circ g$ and $g \circ f$ for the following functions. $f(x) = x^2,\ g(x) = \sqrt{x}$

Review Answers

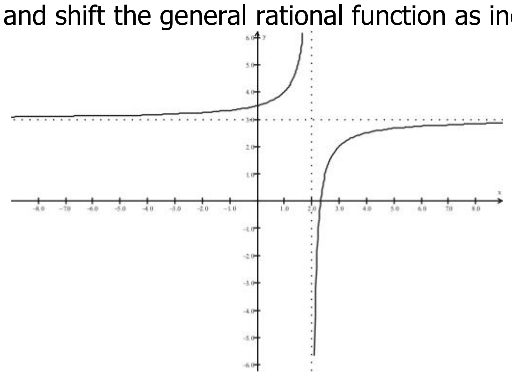
1. The relationship is a function. Domain is All Real Numbers and range $= \{-2 \leq y \leq 2\}$.
2. The relationship is not a function.
3. The domain is $\{x \neq -1, 1\}$. Use the graph and view the table of solutions to determine the range. Using your graphing calculator, enter the relationship on the **Y=** menu. Graph/table shows that range = $(-\infty, 0] \cup (3, \infty)$.



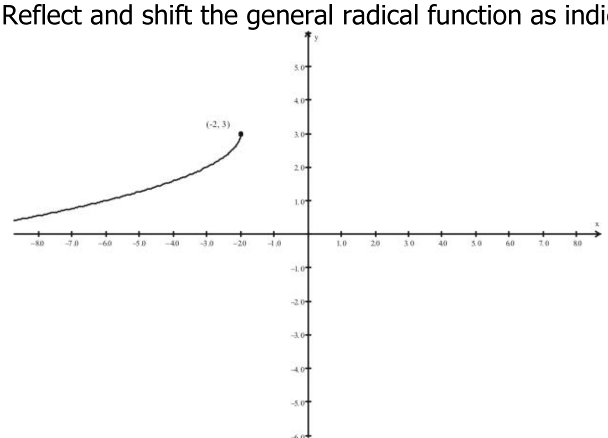
4. Domain = $(-\infty, 3]$; Range = $[0, \infty)$
5. This is the basic absolute value function shifted 3/2 units to the right and down two units. Domain is All Real Numbers and range is $\{y \geq -2\}$.
6. Reflect and shift the general quadratic function as indicated here:



7. Reflect and shift the general rational function as indicated here:



8. Reflect and shift the general radical function as indicated here:



9. $f \circ g = -3\sqrt{x} + 2, g \circ f = \sqrt{-3x + 2}.$
10. $f \circ g = x, g \circ f = x$; any functions where $f \circ g = x, g \circ f = x$ are called inverses; in this problem f and g are inverses of one another. Note that the domain for $f \circ g$ is restricted to only positive numbers and zero.

1.3 Models and Data

Learning Objectives

A student will be able to:

- Fit data to linear models.
- Fit data to quadratic models.
- Fit data to trigonometric models.
- Fit data to exponential growth and decay models.

Introduction

In our last lesson we examined functions and learned how to classify and sketch functions. In this lesson we will use some classic functions to model data. The lesson will be a set of examples of each of the models. For each, we will make extensive use of the graphing calculator.

Let's do a quick review of how to model data on the graphing calculator.

Enter Data in Lists

Press **[STAT]** and then **[EDIT]** to access the lists, **L1 - L6**.

View a Scatter Plot

Press **2nd [STAT PLOT]** and choose accordingly.

Then press **[WINDOW]** to set the limits of the axes.

Compute the Regression Equation

Press **[STAT]** then choose **[CALC]** to access the regression equation menu. Choose the appropriate regression equation (Linear, Quad, Cubic, Exponential, Sine).

Graph the Regression Equation Over Your Scatter Plot

Go to **Y**=> **[MENU]** and clear equations. Press **[VARS]**, then enter $5$and **EQ** and press **[ENTER]** (This series of entries will copy the regression equation to your **Y =** screen.) Press **[GRAPH]** to view the regression equation over your scatter plot

Plotting and Regression in Excel

You can also do regression in an Excel spreadsheet. To start, copy and paste the table of data into Excel. With the two columns highlighted, including the column headings, click on the **Chart** icon and select **XY scatter**. Accept the defaults until a graph appears. Select the graph, then click **Chart**, then **Add Trendline**. From the choices of trendlines choose **Linear**.

Now let's begin our survey of the various modeling situations.

Linear Models

For these kinds of situations, the data will be modeled by the classic linear equation $y = mx + b$.Our task will be to find appropriate values of $m$and $b$for given data.

Example 1:

It is said that the height of a person is equal to his or her wingspan (the measurement from fingertip to fingertip when your arms are stretched horizontally). If this is true, we should be able to take a table of measurements, graph the measurements in an $x - y$coordinate system, and verify this relationship. What kind of graph would you expect to see? ***(Answer: You would expect to see the points on the line*** $y = x$.)

Suppose you measure the height and wingspans of nine of your classmates and gather the following data. Use your graphing calculator to see if the following measurements fit this linear model (the line $y = x$).

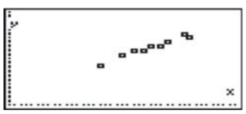
Height (inches)	*Wingspan (inches)*
67	65
64	63
56	57
60	61
62	63

Height (inches)	Wingspan (inches)
71	70
72	69
68	67
65	65

We observe that only one of the measurements has the condition that they are equal. Why aren't more of the measurements equal to each other? ***(Answer: The data do not always conform to exact specifications of the model. For example, measurements tend to be loosely documented so there may be an error arising in the way that measurements were taken.)***

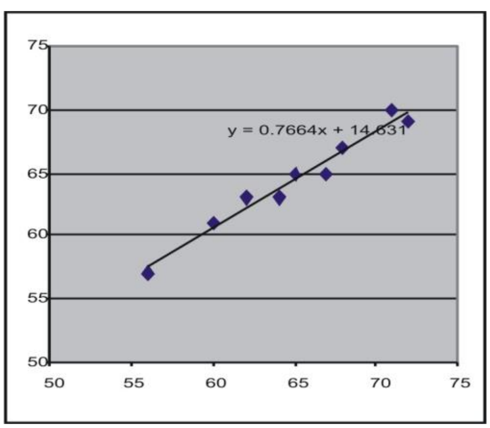
We enter the data in our calculator in **L1** and **L2**. We then view a scatter plot. (Caution: note that the data ranges exceed the viewing window range of $[-10, 10]$. Change the window ranges accordingly to include all of the data, say $[40, 80]$.)

Here is the scatter plot:



Now let us compute the regression equation. Since we expect the data to be linear, we will choose the **linear regression** option from the menu. We get the equation $y = .76x + 14.$

In general we will always wish to graph the regression equation over our data to see the goodness of fit. Doing so yields the following graph, which was drawn with Excel:



Since our calculator will also allow for a variety of non-linear functions to be used as models, we can therefore examine quite a few real life situations. We will first consider an example of quadratic modeling.

Quadratic Models

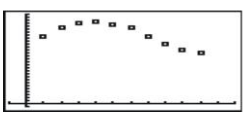
Example 2:

The following table lists the number of Food Stamp recipients (in millions) for each year after 1990.

Years after 1990	*Participants*
1	22.6
2	25.4
3	27.0
4	27.5
5	26.6
6	2.55
7	22.5
8	19.8
9	18.2
10	17.2

We enter the data in our calculator in **L3** and **L4** (that enables us to save the last example's data). We then will view a scatter plot. Change the window ranges accordingly to include all of the data. Use $[-2, 10]$for $x$and $[-2, 30]$for y.

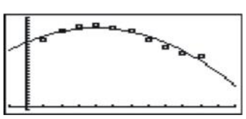
Here is the scatter plot:



Now let us compute the regression equation. Since our scatter plot suggests a quadratic model for the data, we will choose **Quadratic Regression** from the menu. We get the equation:

$$y = -0.30x^2 + 2.38x + 21.67.$$

Let's graph the equation over our data. We see the following graph:



Trigonometric Models

The following example shows how a trigonometric function can be used to model data.

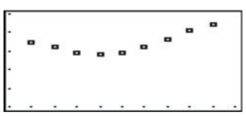
Example 3:

With the skyrocketing cost of gasoline, more people have looked to mass transit as an option for getting around. The following table uses data from the *American Public Transportation Association* to show the number of mass transit trips (in billions) between 1992 and 2000.

Year	*Trips (billions)*
1992	8.5
1993	8.2
1994	7.93
1995	7.8
1996	7.87
1997	8.23
1998	8.6
1999	9.08
2000	9.4

We enter the data in our calculator in **L5** and **L6**. We then will view a scatter plot. Change the window ranges accordingly to include all of the data. Use $[-2, 10]$ for both x and y ranges.

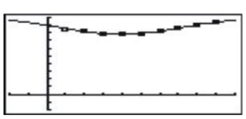
Here is the scatter plot:



Now let us compute the regression equation. Since our scatter plot suggests a sine model for the data, we will choose **Sine Regression** from the menu. We get the equation:

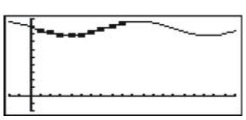
$$y = .9327 * \sin(.4681x + 2.8734) + 8.7358.$$

Let us graph the equation over our data. We see the following graph:



This example suggests that the sine over time t is a function that is used in a variety of modeling situations.

Caution: Although the fit to the data appears quite good, do we really expect the number of trips to continue to go up and down in the future? Probably not. Here is what the graph looks like when projected an additional ten years:



Exponential Models

Our last class of models involves exponential functions. Exponential models can be used to model growth and decay situations. Consider the following data about the declining number of farms for the years 1980 - 2005.

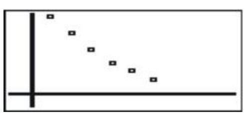
Example 4:

The number of dairy farms has been declining over the past $20+$years. The following table charts the decline:

Year	*Farms (thousands)*
1980	334
1985	269
1990	193
1995	140
2000	105
2005	67

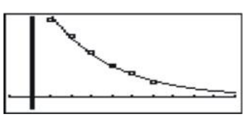
We enter the data in our calculator in **L5** (again entering the years as 1, 2, 3...) and **L6**. We then will view a scatter plot. Change the window ranges accordingly to include all of the data. For the large $y-$values, choose the range $[-50, 350]$ with a scale of $25.$

Here is the scatter plot:



Now let us compute the regression equation. Since our scatter plot suggests an exponential model for the data, we will choose **Exponential Regression** from the menu. We get the equation: $y = 490.6317 * .7266^x$

Let’s graph the equation over our data. We see the following graph:



In the homework we will practice using our calculator extensively to model data.

Lesson Summary

1. Fit data to linear models.
2. Fit data to quadratic models.
3. Fit data to trigonometric models.
4. Fit data to exponential growth and decay models.

Review Questions

1. Consider the following table of measurements of circular objects:

a. Make a scatter plot of the data.
b. Based on your plot, which type of regression will you use?
c. Find the line of best fit.
d. Comment on the values of m and b in the equation.

Object	*Diameter (cm)*	*Circumference (cm)*
Glass	8.3	26.5
Flashlight	5.2	16.7
Aztec calendar	20.2	61.6
Tylenol bottle	3.4	11.6
Popcorn can	13	41.4
Salt shaker	6.3	20.1
Coffee canister	11.3	35.8
Cat food bucket	33.5	106.5
Dinner plate	27.3	85.6
Ritz cracker	4.9	15.5

2. Manatees are large, gentle sea creatures that live along the Florida coast. Many manatees are killed or injured by power boats. Here are data on powerboat registrations (in thousands) and the number of manatees killed by boats in Florida from 1987 - 1997.
   a. Make a scatter plot of the data.
   b. Use linear regression to find the line of best fit.
   c. Suppose in the year 2000, powerboat registrations increase to 700,000. Predict how many manatees will be killed. Assume a linear model and find the line of best fit.

Year	*Boats*	*Manatees killed*
1987	447	13
1988	460	21
1989	480	24
1990	497	16
1991	512	24
1992	513	21
1993	526	15

Year	Boats	Manatees killed
1994	557	33
1995	585	34
1996	614	34
1997	645	39

3. A passage in ***Gulliver's Travels*** states that the measurement of "Twice around the wrist is once around the neck." The table below contains the wrist and neck measurements of 10 people.
   a. Make a scatter plot of the data.
   b. Find the line of best fit and comment on the accuracy of the quote from the book.
   c. Predict the distance around the neck of Gulliver if the distance around his wrist is found to be 52 cm.

Wrist (cm)	Neck (cm)
17.9	39.5
16	32.5
16.5	34.7
15.9	32
17	33.3
17.3	32.6
16.8	33
17.3	31.6
17.7	35
16.9	34

4. The following table gives women's average percentage of men's salaries for the same jobs for each 5-year period from 1960 - 2005.
   a. Make a scatter plot of the data.
   b. Based on your sketch, should you use a linear or quadratic model for the data?
   c. Find a model for the data.
   d. Can you explain why the data seems to dip at first and then grow?

Year	Percentage
1960	42
1965	36

Year	Percentage
1970	30
1975	37
1980	41
1985	42
1990	48
1995	55
2000	58
2005	60

5. Based on the model for the previous problem, when will women make as much as men? Is your answer a realistic prediction?
6. The average price of a gallon of gas for selected years from 1975 - 2008 is given in the following table:
    a. Make a scatter plot of the data.
    b. Based on your sketch, should you use a linear, quadratic, or cubic model for the data?
    c. Find a model for the data.
    d. If gas continues to rise at this rate, predict the price of gas in the year 2012.

Year	Cost
1975	1
1976	1.75
1981	2
1985	2.57
1995	2.45
2005	2.75
2008	3.45

7. For the previous problem, use a linear model to analyze the situation. Does the linear method provide a better estimate for the predicted cost for the year 2011? Why or why not?
8. Suppose that you place $1,000 in a bank account where it grows exponentially at a rate of 12% continuously over the course of five years. The table below shows the amount of money you have at the end of each year.
    a. Find the exponential model.
    b. In what year will you triple your original amount?

Year	Amount

Year	Amount
0	1000
1	1127.50
2	1271.24
3	1433.33
4	1616.07
5	1822.11
6	2054.43

9. Suppose that in the previous problem, you started with $3,000 but maintained the same interest rate.
   a. Give a formula for the exponential model. (Hint: note the coefficient and exponent in the previous answer!)
   b. How long will it take for the initial amount, $3,000, to triple? Explain your answer.
10. The following table gives the average daily temperature for Indianapolis, Indiana for each month of the year.
    a. Construct a scatter plot of the data.
    b. Find the sine model for the data.

Month	Avg Temp (F)
Jan	22
Feb	26.3
March	37.8
April	51
May	61.7
June	75.3
July	78.5
Aug	84.3
Sept	68.5
Oct	53.2
Nov	38.7
Dec	26.6

Review Answers

1.
    a. .
    b. Linear.
    c. $y = 3.1334x + .3296.$
    d. m is an estimate of π, and b should be zero but due to error in measurement it is not.
2.
    a. .
    b. $y = .120546x - 39.0465$
    c. About 46 manatees will be killed in the year 2000. Note: there were actually 81 manatees killed in the year 2000.
3.
    a. .
    b. $y = 2.0131x - 0.2634$
    c. 104.42 cm
4.
    a. .
    b. Quadratic
    c. $y = .4848x^2 - 2.4545x + 39.7333.$
    d. It might be because the first wave of women into the workforce tended to take whatever jobs they could find without regard for salary.
5. The data suggest that women will reach 100% in 2009; this is unrealistic based on current reports that women still lag far behind men in equal salaries for equal work.
6.
    a. .
    b. Cubic.
    c. $y = .0277x^3 - 0.3497x^2 + 1.6203x - 0.3157.$
    d. $12.15
7. Linear $y = 0.35x + 0.88$; Predicted cost in 2012 is $4.73; it is hard to say which model works best but it seems that the use of a cubic model may overestimate the cost in the short term.
8.
    a. $A = 1000 * 2.7182^{.12t}$
    b. the amount will triple early in Year 9.
9.
    a. $A = 3000 * 2.7182^{.12t}.$
    b. The amount will triple early in Year 9 as in the last problem because the exponential equations $3000 = 1000 * 2.7182^{.12t}$ and $9000 = 1000 * 2.7182^{.12t}$ both reduce to the same equation $3 = 2.7182^{.12t}$ and hence have the same solution.
10. $y = 30.07 * \sin(.5196x - 2.1503) + 51.46.$

1.4 The Calculus

Learning Objectives

A student will be able to:

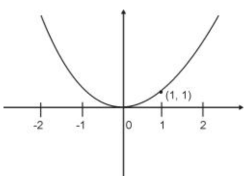
- Use linear approximations to study the limit process.
- Compute approximations for the slope of tangent lines to a graph.
- Introduce applications of differential calculus.

Introduction

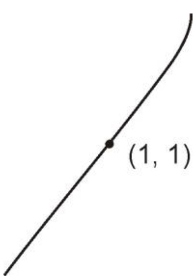
In this lesson we will begin our discussion of the key concepts of calculus. They involve a couple of basic situations that we will come back to time and again throughout the book. For each of these, we will make use of some basic ideas about how we can use straight lines to help approximate functions.

Let's start with an example of a simple function to illustrate each of the situations.

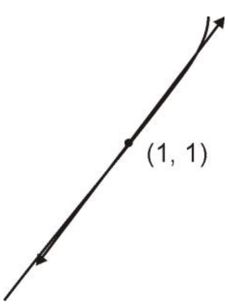
Consider the quadratic function $f(x) = x^2$. We recall that its graph is a parabola. Let's look at the point $(1, 1)$ on the graph.



Suppose we magnify our picture and zoom in on the point $(1, 1)$. The picture might look like this:



We note that the curve now looks very much like a straight line. If we were to overlay this view with a straight line that intersects the curve at $(1, 1)$, our picture would look like this:



We can make the following observations. First, this line would appear to provide a good estimate of the value of $f(x)$ for $x-$values very close to $x = 1$. Second, the approximations appear to be getting closer and closer to the actual value of the function as we take points on the line closer and closer to the point $(1, 1)$. This line is called ***the tangent line to*** $f(x)$ ***at*** $(1, 1)$. This is one of the basic situations that we will explore in calculus.

Tangent Line to a Graph

Continuing our discussion of the tangent line to $f(x)$ at $(1, 1)$, we next wish to find the equation of the tangent line. We know that it passes though $(1, 1)$, but we do not yet have enough information to generate its equation. What other information do we need? ***(Answer: The slope of the line.)***

Yes, we need to find the slope of the line. We would be able to find the slope if we knew a second point on the line. So let's choose a point P on the line, very close to $(1, 1)$. We can approximate the coordinates of P using the function $f(x) = x^2$; hence $P(x, x^2)$. Recall that for points very close to $(1, 1)$, the points on the line are close approximate points of the function. Using this approximation, we can compute the slope of the tangent as follows:

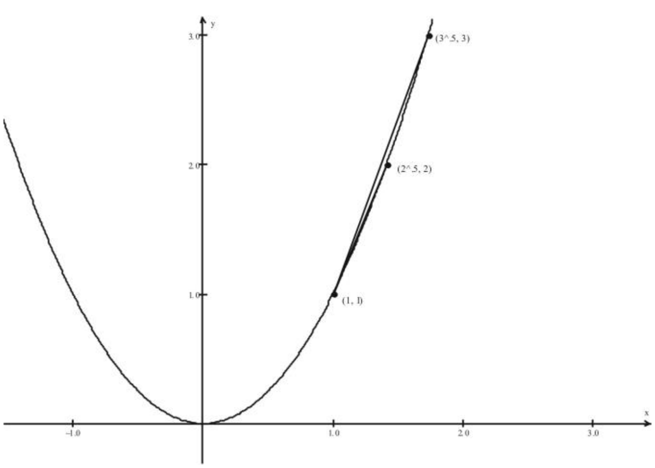
$m = (x^2 - 1)/(x - 1) = x + 1$ (Note: We choose points very close to $(1, 1)$ but not the point itself, so $x \neq 1$).

In particular, for $x = 1.25$ we have $P(1.25, 1.5625)$ and $m = x + 1 = 2.25$. Hence the equation of the tangent line, in point slope form is $y - 1 = 2.25(x - 1)$. We can keep getting closer to the actual value of the slope by taking P closer to $(1, 1)$, or x closer and closer to $x = 1$, as in the following table:

$P(x, y)$	m
$(1.2, 1.44)$	2.2
$(1.15, 1.3225)$	2.15
$(1.1, 1.21)$	2.1
$(1.05, 1.1025)$	2.05
$(1.005, 1.010025)$	2.005
$(1.0001, 1.00020001)$	2.0001

As we get closer to $(1, 1)$, we get closer to the actual slope of the tangent line, the value 2. We call the slope of the tangent line at the point $(1, 1)$ **the derivative of the function** $f(x)$ **at the point** $(1, 1)$.

Let's make a couple of observations about this process. First, we can interpret the process graphically as finding secant lines from $(1, 1)$ to other points on the graph. From the diagram we see a sequence of these secant lines and can observe how they begin to approximate the tangent line to the graph at $(1, 1)$. The diagram shows a pair of secant lines, joining $(1, 1)$ with points $(\sqrt{2}, 2)$ and $(\sqrt{3}, 3)$.

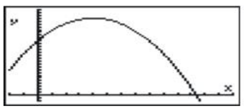


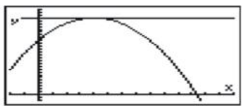
Second, in examining the sequence of slopes of these secants, we are systematically observing ***approximate slopes of the function*** as point P gets closer to $(1, 1)$. Finally, producing the table of slope values above was an inductive process in which we generated some data and then looked to deduce from our data the value to which the generated results tended. In this example, the slope values appear to approach the value 2. This process of finding how function values behave as we systematically get closer and closer to particular $x-$values is the process of finding ***limits***. In the next lesson we will formally define this process and develop some efficient ways for computing limits of functions.

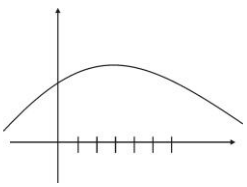
Applications of Differential Calculus

Maximizing and Minimizing Functions

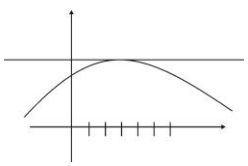
Recall from Lesson 1.3 our example of modeling the number of Food Stamp recipients. The model was found to be $y = -0.5x^2 + 4x + 19$ with graph as follows: (Use viewing window ranges of $[-2, 14]$ on x and $[-2, 30]$ on y)







We note that the function appears to attain a maximum value about an $x-$value somewhere around $x = 4$. Using the process from the previous example, what can we say about the tangent line to the graph for that x value that yields the maximum y value (the point at the top of the parabola)? ***(Answer: the tangent line will be horizontal, thus having a slope of 0.)***



Hence we can use calculus to model situations where we wish to maximize or minimize a particular function. This process will be particularly important for looking at situations from business and industry where polynomial functions provide accurate models.

Velocity of a Falling Object

We can use differential calculus to investigate the velocity of a falling object. Galileo found that the distance traveled by a falling object was proportional to the square of the time it has been falling:

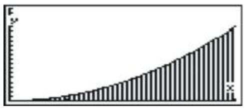
$$s(t) = 4.9t^2.$$

The average velocity of a falling object from $t = a$ to $t = b$ is given by $(s(b) - s(a))/(b - a)$.

HW Problem #10 will give you an opportunity to explore this relationship. In our discussion, we saw how the study of tangent lines to functions yields rich information about functions. We now consider the second situation that arises in Calculus, the central problem of ***finding the area under the curve of a function*** $f(x)$.

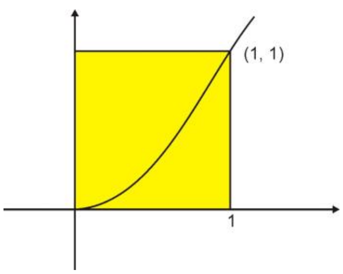
Area Under a Curve

First let's describe what we mean when we refer to the area under a curve. Let's reconsider our basic quadratic function $f(x) = x^2$. Suppose we are interested in finding the area under the curve from $x = 0$ to $x = 1$.

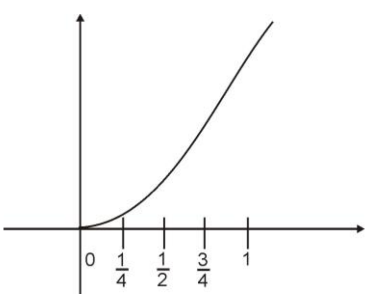


We see the cross-hatched region that lies between the graph and the $x-$axis. That is the area we wish to compute. As with approximating the slope of the tangent line to a function, we will use familiar linear methods to approximate the area. Then we will repeat the iterative process of finding better and better approximations.

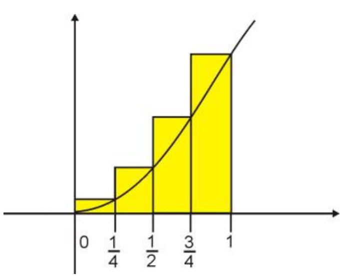
Can you think of any ways that you would be able to approximate the area? (Answer: One ideas is that we could compute the area of the square that has a corner at $(1, 1)$ to be $A = 1$ and then take half to find an area $A = 1/2$. This is one estimate of the area and it is actually a pretty good first approximation.)



We will use a variation of this covering of the region with quadrilaterals to get better approximations. We will do so by dividing the $x-$interval from $x = 0$ to $x = 1$ into equal sub-intervals. Let's start by using four such subintervals as indicated:



We now will construct four rectangles that will serve as the basis for our approximation of the area. The subintervals will serve as the width of the rectangles. We will take the length of each rectangle to be the maximum value of the function in the subinterval. Hence we get the following figure:



If we call the rectangles R1–R4, from left to right, then we have the areas

$$R_1 = \frac{1}{4} * f\left(\frac{1}{4}\right) = \frac{1}{64},$$
$$R_2 = \frac{1}{4} * f\left(\frac{1}{2}\right) = \frac{1}{16},$$
$$R_3 = \frac{1}{4} * f\left(\frac{3}{4}\right) = \frac{9}{64},$$
$$R_4 = \frac{1}{4} * f(1) = \frac{1}{4},$$

and $R_1 + R_2 + R_3 + R_4 = \frac{30}{64} = \frac{15}{32}.$

Note that this approximation is very close to our initial approximation of $1/2.$ However, since we took the maximum value of the function for a side of each rectangle, this process tends to overestimate the true value. We could have used the minimum value of the function in each sub-interval. Or we could have used the value of the function at the midpoint of each sub-interval.

Can you see how we are going to improve our approximation using successive iterations like we did to approximate the slope of the tangent line? ***(Answer: we will sub-divide the interval from*** $x = 0$ ***to*** $x = 1$ ***into more and more sub-intervals, thus creating successively smaller and smaller rectangles to refine our estimates.)***

Example 1:

The following table shows the areas of the rectangles and their sum for rectangles having width $w = 1/8.$

Rectangle R_i	Area of R_i
R_1	$\frac{1}{512}$
R_2	$\frac{4}{512}$
R_3	$\frac{9}{512}$
R_4	$\frac{16}{512}$
R_5	$\frac{25}{512}$
R_6	$\frac{36}{512}$
R_7	$\frac{49}{512}$
R_8	$\frac{64}{512}$

$A = \sum R_i = \frac{195}{512}$. This value is approximately equal to $.3803.$ Hence, the approximation is now quite a bit less than $.5.$ For sixteen rectangles, the value is $\frac{1432}{4096}$ which is approximately equal to $.34.$ Can you guess what the true area will approach? ***(Answer: using our successive approximations, the area will approach the value*** $1/3.$***)***

We call this process of finding the area under a curve ***integration of*** $f(x)$ ***over the interval*** $[0, 1].$

Applications of Integral Calculus

We have not yet developed any computational machinery for computing ***derivatives*** and ***integrals*** so we will just state one popular application of integral calculus that relates the derivative and integrals of a function.

Example 2:

There are quite a few applications of calculus in business. One of these is the cost function $C(x)$ of producing x items of a product. It can be shown that the derivative of the cost function that gives the slope of the tangent line is another function that that gives the cost to produce an additional unit of the product. This is called the ***marginal cost*** and is a very important piece of information for management to have. Conversely, if one knows the marginal cost as a function of x, then finding the area under the curve of the function will give back the cost function $C(x)$.

Lesson Summary

1. We used linear approximations to study the limit process.
2. We computed approximations for the slope of tangent lines to a graph.
3. We analyzed applications of differential calculus.
4. We analyzed applications of integral calculus.

Review Questions

1. For the function $f(x) = x^2$ approximate the slope of the tangent line to the graph at the point $(3, 9)$. Use the following set of x-values to generate the sequence of secant line slopes: $x = 2.9, 2.95, 2.975, 2.995, 2.999.$ What value does the sequence of slopes approach?
2. Consider the function $f(x) = x^2$.
   a. For what values of x would you expect the slope of the tangent line to be negative?
   b. For what value of x would you expect the tangent line to have slope m = 0?
   c. Give an example of a function that has two different horizontal tangent lines.
3. Consider the function $p(x) = x^3 - x$. Generate the graph of $p(x)$ using your calculator.
   a. Approximate the slope of the tangent line to the graph at the point $(2, 6)$. Use the following set of x-values to generate the sequence of secant line slopes. $x = 2.1, 2.05, 2.005, 2.001, 2.0001.$
   b. For what values of x do the tangent lines appear to have slope of 0? (Hint: Use the calculate function in your calculator to approximate the x-values.)
   c. For what values of x do the tangent lines appear to have positive slope?
   d. For what values of x do the tangent lines appear to have negative slope?
4. The cost of producing x Hi-Fi stereo receivers by Yamaha each week is modeled by the following function:

   $C(x) = 850 + 200x - .3x^2.$

   a. Generate the graph of C(x) using your calculator. (Hint: Change your viewing window to reflect the high y values.)
   b. For what number of units will the function be maximized?
   c. Estimate the slope of the tangent line at $x = 200, 300, 400.$
   d. Where is marginal cost positive?
5. Find the area under the curve of $f(x) = x^2$ from x = 1 to x = 3. Use a rectangle method that uses the minimum value of the function within sub-intervals. Produce the approximation for each case of the subinterval cases.
   a. four sub-intervals.
   b. eight sub-intervals.
   c. Repeat part a. using a Mid-Point Value of the function within each sub-interval.
   d. Which of the answers in a. - c. provide the best estimate of the actual area?
6. Consider the function $p(x) = -x^3 + 4x$.

a. Find the area under the curve from x = 0 to x = 1.
b. Can you find the area under the curve from x = -1 to x = 0? Why or why not? What is problematic for this computation?

7. Find the area under the curve of $f(x) = \sqrt{x}$ from x = 1 to x = 4. Use the Max Value rectangle method with six sub-intervals to compute the area.
8. The Eiffel Tower is 320 meters high. Suppose that you drop a ball off the top of the tower. The distance that it falls is a function of time and is given by $s(t) = 4.9t^2$. Find the velocity of the ball after 4 seconds. (Hint: the average velocity for a time interval is **average velocity = change in distance/change in time**. Investigate the average velocity for t intervals close to t = 4 such as $3.9 \le t \le 4$ and closer and see if a pattern is evident.)

Review Answers

1. *m* = 6
2.
   a. For $x < 0$
   b. At $x = 0$ the tangent line is horizontal and thus has slope of 0.
   c. Many different examples; for instance, a polynomial function such as $p(x) = x^3 - 4x$.
3.
   a. Slope tends toward $m = 11$.
   b. $x = \pm .57$
   c. $(-\infty, -.57) \cup (.57, \infty)$
   d. $(-.57, .57)$
4.
   a. 80, 20, -40
   b. 333 units
   c. 80, 20, -40
   d. x < 333
5.
   a. 6.75
   b. 7.6875
   c. 8.625
   d. Part c.
6.
   a. 1.75
   b. The graph drops below the x-axis into the third quadrant. Hence we are not finding the area below the curve but actually the area between the curve and the x-axis. But note that the curve is symmetric about the origin. Hence the region from x = -1 to x = 0 will have the same area as the region from x = 0 to x = 1.
7. 4.911
8. 39.2 m/s

1.5 Finding Limits

Learning Objectives

A student will be able to:

- Find the limit of a function numerically.
- Find the limit of a function using a graph.
- Identify cases when limits do not exist.
- Use the formal definition of a limit to solve limit problems.

Introduction

In this lesson we will continue our discussion of the limiting process we introduced in Lesson 1.4. We will examine numerical and graphical techniques to find limits where they exist and also to examine examples where limits do not exist. We will conclude the lesson with a more precise definition of limits.

Let’s start with the notation that we will use to denote limits. We indicate the limit of a function as the $x$values approach a particular value of x, say a, as

$$\lim_{x\to a} f(x).$$

So, in the example from Lesson 1.3 concerning the function $f(x)=x^2$, we took points that got closer to the point on the graph $(1,1)$and observed the sequence of slope values of the corresponding secant lines. Using our limit notation here, we would write

$$\lim_{x\to 1}\frac{x^2-1}{x-1}.$$

Recall also that we found that the slope values tended to the value $x=2$; hence using our notation we can write

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2.$$

Finding Limits Numerically

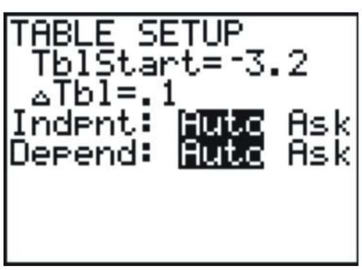
In our example in Lesson 1.3 we used this approach to find that $\lim_{x\to 1}\frac{x^2-1}{x-1}=2$. Let’s apply this technique to a more complicated function.

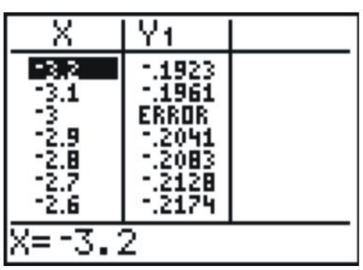
Consider the rational function $f(x)=\frac{x+3}{x^2+x-6}$. Let’s find the following limit:

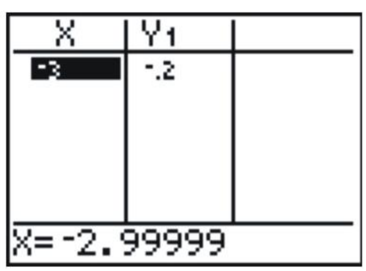
$$\lim_{x\to -3}\frac{x+3}{x^2+x-6}.$$

Unlike our simple quadratic function, $f(x)=x^2$, it is tedious to compute the points manually. So let’s use the **[TABLE]** function of our calculator. Enter the equation in your calculator and examine the table of points of the function. Do you notice anything unusual about the points? ***(Answer: There are error readings indicated for*** $x=-3,2$ ***because the function is not defined at these values.)***

Even though the function is not defined at $x=-3$, we can still use the calculator to read the $y-$values for $x$values very close to $x=-3$. Press **2ND [TBLSET]** and set **Tblstart** to $-3.2$and $\triangle$to 0.1(see screen on left below). The resulting table appears in the middle below.







Can you guess the value of $\lim_{x\to -3}\frac{x+3}{x^2+x-6}$? If you guessed $-.20=-(1/5)$you would be correct. Before we finalize our answer, let’s get even closer to $x=-3$and determine its function value using the **[CALC VALUE]** tool.

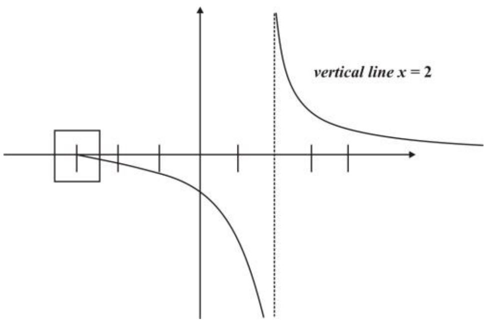
Press **2ND [TBLSET]** and change **Indpnt** from **Auto** to **Ask**. Now when you go to the table, enter $x = -2.99999$.and press **[ENTER]** and you will see the screen on the right above. Press **[ENTER]** and see that the function value is $x = -0.2$.which is the closest the calculator can display in the four decimal places allotted in the table. So our guess is correct and $\lim_{x \to -3} \frac{x+3}{x^2+x-6} = -\frac{1}{5}$.

Finding Limits Graphically

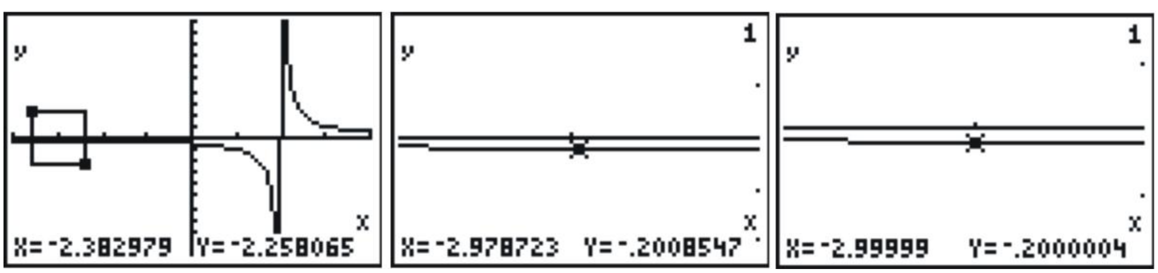
Let's continue with the same problem but now let's focus on using the graph of the function to determine its limit.

$$\lim_{x \to -3} \frac{x+3}{x^2+x-6}$$

We enter the function in the $Y =$menu and sketch the graph. Since we are interested in the value of the function for x close to $x = -3$.we will look to **[ZOOM]** in on the graph at that point.



Our graph above is set to the normal viewing window $[-10, 10]$.Hence the values of the function appear to be very close to 0. But in our numerical example, we found that the function values approached $-.20 = -(1/5)$.To see this graphically, we can use the **[ZOOM]** and **[TRACE]** function of our calculator. Begin by choosing **[ZOOM]** function and choose **[BOX]**. Using the directional arrows to move the cursor, make a box around the $x$value -3.(See the screen on the left below Press **[ENTER]** and **[TRACE]** and you will see the screen in the middle below.) In **[TRACE]** mode, type the number $-2.99999$and press **[ENTER]**. You will see a screen like the one on the right below.

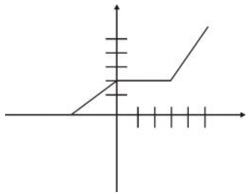


The graphing calculator will allow us to calculate limits graphically, provided that we have the function rule for the function so that we can enter its equation into the calculator. What if we have only a graph given to us and we are asked to find certain limits?

It turns out that we will need to have pretty accurate graphs that include sufficient detail about the location of data points. Consider the following example.

Example 1:

Find $\lim_{x \to 3} f(x)$for the function pictured here. Assume units of value 1 for each unit on the axes.



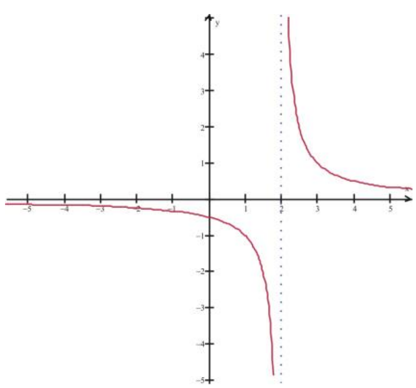
By inspection, we see that as we approach the value $x = 3$from the left, we do so along what appears to be a portion of the horizontal line $y = 2$.We see that as we approach the value $x = 3$from the right, we do so along a line segment having positive slope. In either case, the $y$values of $f(x)$approaches $y = 2$.

Nonexistent Limits

We sometimes have functions where $\lim_{x \to a} f(x)$does not exist. We have already seen an example of a function where our a value was not in the domain of the function. In particular, the function was not defined for $x = -3, 2$.but we could still find the limit as $x \to -3$.

$$\lim_{x \to -3} \frac{x+3}{x^2+x-6} = -\frac{1}{5}$$

What do you think the limit will be as we let $x \to 2$?



$$\lim_{x \to 2} \frac{x+3}{x^2+x-6}$$

Our inspection of the graph suggests that the function around $x = 2$does not appear to approach a particular value. For $x > 2$,the points all lie in the first quadrant and appear to grow very quickly to large positive numbers as we get close to $x = 2$.Alternatively, for $x < 2$we see that the points all lie in the fourth quadrant and decrease to large negative numbers. If we inspect actual values very close to $x = 2$we can see that the values of the function do not approach a particular value.

x	y
1.999	− 1000
1.9999	− 10000
2	ERROR
2.001	1000
2.0001	10000

For this example, we say that $\lim_{x \to 2} \frac{x+3}{x^2+x-6}$ does not exist.

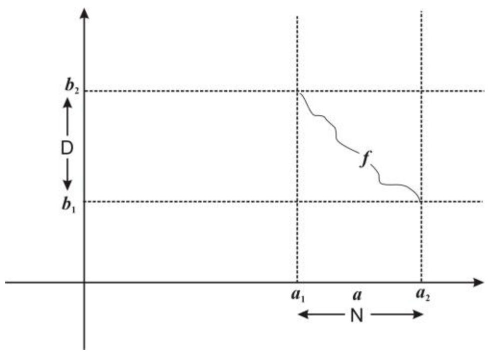
Formal Definition of a Limit

We conclude this lesson with a formal definition of a limit.

Definition:

This definition is somewhat intuitive to us given the examples we have covered. Geometrically, the definition means that for any lines $y = b_1, y = b_2$ below and above the line $y = L$, there exist vertical lines $x = a_1, x = a_2$ to the left and right of $x = a$ so that the graph of $f(x)$ between $x = a_1$ and $x = a_2$ lies between the lines $y = b_1$ and $y = b_2$. The key phrase in the above statement is "for every open interval D", which means that even if D is very, very small (that is, $f(x)$ is very, very close to L), it still is possible to find interval N where $f(x)$ is defined for all values except possibly $x = a$.

We say that the limit of a function $f(x)$ at a is L, written as $\lim_{x\to a} f(x) = L$, if for every open interval D of L, there exists an open interval N of a, that does not include a, such that $f(x)$ is in D for every x in N.



Example 2:

Use the definition of a limit to prove that

$$\lim_{x\to 3}(2x+1) = 7.$$

We need to show that for each open interval of 7, we can find an open neighborhood of 3, that does not include 3, so that all x in the open neighborhood map into the open interval of 7.

Equivalently, we must show that for every interval of 7, say $(7-\varepsilon, 7+\varepsilon)$, we can find an interval of 3, say $(3-\delta, 3+\delta)$, such that $(7-\varepsilon < 2x+1 < 7+\varepsilon)$ whenever $(3-\delta < x < 3+\delta)$.

The first inequality is equivalent to $6-\varepsilon < 2x < 6+\varepsilon$ and solving for x, we have

$$3 - \frac{\varepsilon}{2} < x < 3 + \frac{\varepsilon}{2}.$$

Hence if we take $\delta = \frac{\varepsilon}{2}$, we will have $3-\delta < x < 3+\delta \Rightarrow 7-\varepsilon < 2x+1 < 7+\varepsilon.$

Fortunately, we do not have to do this to evaluate limits. In Lesson 1.6 we will learn several rules that will make the task manageable.

Lesson Summary

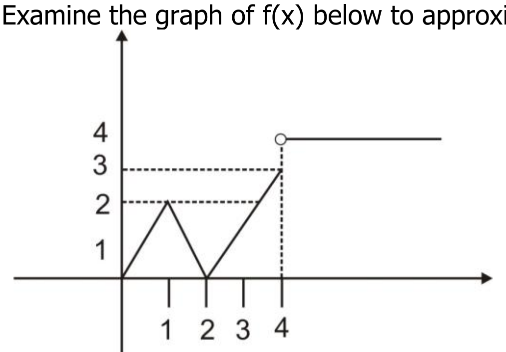
1. We learned to find the limit of a function numerically.
2. We learned to find the limit of a function using a graph.
3. We identified cases when limits do not exist.
4. We used the formal definition of a limit to solve limit problems.

Multimedia Links

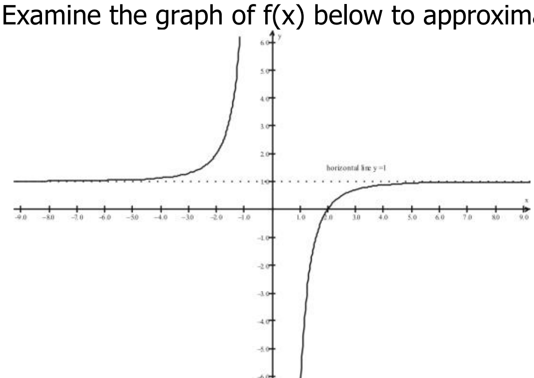
For another look at the definition of a limit, the series of videos at Tutorials for the Calculus Phobe has a nice intuitive introduction to this fundamental concept (despite the whimsical name). If you want to experiment with limits yourself, follow the sequence of activities using a graphing applet at Informal Limits. Directions for using the graphing applets at this very useful site are also available at Applet Intro.

Review Questions

1. Use a table of values to find $\lim_{x\to -2} \frac{x^2-4}{x+2}$. Use x-values of x = -1.9, -1.99, -1.999, -2.1, -2.01, -2.001. What value does the sequence of values approach?
2. Use a table of values to find $\lim_{x\to \frac{1}{2}} \frac{2x-1}{2x^2+3x-2}$. Use $x-$values of x = .49, .499, .4999, .51, .501, .5001. What value does the sequence of values approach?
3. Consider the function $p(x) = 3x^3 - 3x.$ Generate the graph of p(x) using your calculator. Find each of the following limits if they exist. Use tables with appropriate $x$values to determine the limits.
   a. $\lim_{x\to 4}(3x^3 - 3x)$
   b. $\lim_{x\to -4}(3x^3 - 3x)$
   c. $\lim_{x\to 0}(3x^3 - 3x)$
   d. Find the values of the function corresponding to x = 4, -4, 0. How do these function values compare to the limits you found in #a-c? Explain your answer.
4. Examine the graph of f(x) below to approximate each of the following limits if they exist.



   a. $\lim_{x\to 3} f(x)$
   b. $\lim_{x\to 2} f(x)$
   c. $\lim_{x\to 1} f(x)$
   d. $\lim_{x\to 4} f(x)$
5. Examine the graph of f(x) below to approximate each of the following limits if they exist.



   a. $\lim_{x\to 2} f(x)$
   b. $\lim_{x\to 0} f(x)$
   c. $\lim_{x\to 4} f(x)$
   d. $\lim_{x\to 50} f(x)$

In problems #6-8, determine if the indicated limit exists, and if so, find the limit. Provide a numerical argument to justify your answer.

6. $\lim_{x \to 2}(x^2 + 3)$
7. $\lim_{x \to -1} \frac{x+1}{x^2-1}$
8. $\lim_{x \to 2} \sqrt{-2x + 5}$

In problems #9-10, determine if the indicated limit exists. Provide a graphical argument to justify your answer. (Hint: Make use of the **[ZOOM]** and **[TABLE]** functions of your calculator to view functions values close to the indicated x value.

9. $\lim_{x \to 4}(x^2 + 3x)$
10. $\lim_{x \to -1} \frac{|x+1|}{x+1}$

Review Answers

1. -4
2. 2/5
3.
    a. lim as x -> 4 of ($3x^3$ - 3x) = 180
    b. lim as x -> -4 of ($3x^3$ - 3x) = -180
    c. $\lim_{x \to 0}(3x^3 - 3x) = 0$
    d. They are the same values because the function is defined for each of these x-values.
4.
    a. $\lim_{x \to 3} f(x) = 1.5$
    b. $\lim_{x \to 2} f(x) = 0$
    c. $\lim_{x \to 1} f(x) = 2$
    d. $\lim_{x \to 4} f(x)$ does not exist.
5.
    a. $\lim_{x \to 2} f(x) = 0$
    b. $\lim_{x \to 0} f(x)$ does not exist.
    c. $\lim_{x \to 4} f(x)$ is some number close to 1 and less than 1, but not equal to 1.
    d. $\lim_{x \to 50} f(x)$ is some number close to 1 and less than 1, but not equal to 1.
6. The limit does exist. This can be verified by using the **[TRACE]** or **[TABLE]** function of your calculator, applied to x values very close to $x = 2$. The limit is 7.
7. The limit does exist. This can be verified by using the **[TRACE]** or **[TABLE]** function of your calculator, applied to x values very close to $x = -1$. The limit is -1/2.
8. The limit does exist. This can be verified by using the **[TRACE]** or **[TABLE]** function of your calculator, applied to x values very close to $x = 2$. The limit is 1.
9. The limit does exist. This can be verified with either the **[TRACE]** or **[TABLE]** function of your calculator.
10. The limit does not exist; **[ZOOM]** in on the graph around x = -1 and see that the y-values approach a different value when approached from the right and from the left.

1.6 Evaluating Limits

Learning Objectives

A student will be able to:

- Find the limit of basic functions.
- Use properties of limits to find limits of polynomial, rational and radical functions.
- Find limits of composite functions.
- Find limits of trigonometric functions.
- Use the Squeeze Theorem to find limits.

Introduction

In this lesson we will continue our discussion of limits and focus on ways to evaluate limits. We will observe the limits of a few basic functions and then introduce a set of laws for working with limits. We will conclude the lesson with a theorem that will allow us to use an indirect method to find the limit of a function.

Direct Substitution and Basic Limits

Let's begin with some observations about limits of basic functions. Consider the following limit problems:

$$\lim_{x\to 2} 5,$$
$$\lim_{x\to 4} x.$$

These are examples of limits of basic constant and linear functions, $f(x) = c$ and $f(x) = mx + b.$

We note that each of these functions are defined for all real numbers. If we apply our techniques for finding the limits we see that

$$\lim_{x\to 2} 5 = 5,$$
$$\lim_{x\to 4} x = 4,$$

and observe that for each the limit equals the value of the function at the $x-$value of interest:

$$\lim_{x\to 2} 5 = f(5) = 5,$$
$$\lim_{x\to 4} x = f(4) = 4.$$

Hence $\lim_{x\to a} f(x) = f(a)$. This will also be true for some of our other basic functions, in particular all polynomial and radical functions, provided that the function is defined at $x = a$. For example, $\lim_{x\to 3} x^3 = f(3) = 27$ and $\lim_{x\to 4} \sqrt{x} = f(4) = 2$. The properties of functions that make these facts true will be discussed in Lesson 1.7. For now, we wish to use this idea for evaluating limits of basic functions. However, in order to evaluate limits of more complex function we will need some properties of limits, just as we needed laws for dealing with complex problems involving exponents. A simple example illustrates the need we have for such laws.

Example 1:

Evaluate $\lim_{x\to 2}(x^3 + \sqrt{2x})$. The problem here is that while we know that the limit of each individual function of the sum exists, $\lim_{x\to 2} x^2 = 8$ and $\lim_{x\to 2} \sqrt{2x} = 2$, our basic limits above do not tell us what happens when we find the limit of a sum of functions. We will state a set of properties for dealing with such sophisticated functions.

Properties of Limits

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then

1. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ where c is a real number,
2. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ where n is a real number,
3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x),$
4. $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x),$
5. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided that $\lim_{x\to a} g(x) \neq 0.$

With these properties we can evaluate a wide range of polynomial and radical functions. Recalling our example above, we see that

$$\lim_{x\to 2}(x^3+\sqrt{2x}) = \lim_{x\to 2}(x^3) + \lim_{x\to 2}(\sqrt{2x}) = 8+2 = 10.$$

Find the following limit if it exists:

$$\lim_{x\to -4}(2x^2-\sqrt{-x}).$$

Since the limit of each function within the parentheses exists, we can apply our properties and find

$$\lim_{x\to -4}(2x^2-\sqrt{-x}) = \lim_{x\to -4}2x^2 - \lim_{x\to -4}\sqrt{-x}.$$

Observe that the second limit, $\lim_{x\to -4}\sqrt{-x}$, is an application of Law #2 with $n=\frac{1}{2}$. So we have $\lim_{x\to -4}(2x^2-\sqrt{-x}) = \lim_{x\to -4}2x^2 - \lim_{x\to -4}\sqrt{-x} = 32-2 = 30.$

In most cases of sophisticated functions, we simplify the task by applying the Properties as indicated. We want to examine a few exceptions to these rules that will require additional analysis.

Strategies for Evaluating Limits of Rational Functions

Let's recall our example

$$\lim_{x\to 1}\frac{x^2-1}{x-1}.$$

We saw that the function did not have to be defined at a particular value for the limit to exist. In this example, the function was not defined for $x=1$. However we were able to evaluate the limit numerically by checking functional values around $x=1$and found $\lim_{x\to 1}\frac{x^2-1}{x-1}=2$.

Note that if we tried to evaluate by direct substitution, we would get the quantity $0/0$, which we refer to as an ***indeterminate form.*** In particular, Property #5 for finding limits does not apply since $\lim_{x\to 1}(x-1)=0$. Hence in order to evaluate the limit without using numerical or graphical techniques we make the following observation. The numerator of the function can be factored, with one factor common to the denominator, and the fraction simplified as follows:

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1.$$

In making this simplification, we are indicating that the original function can be viewed as a linear function for $x$values close to but not equal to 1, that is,

$\frac{x^2-1}{x-1}=x+1$for $x\neq 1$. In terms of our limits, we can say

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 1+1 = 2.$$

Example 2:

Find $\lim_{x\to 0}\frac{x^2+5x}{x}$.

This is another case where direct substitution to evaluate the limit gives the indeterminate form $0/0$. Reducing the fraction as before gives:

$$\lim_{x\to 0}\frac{x^2+5x}{x}=\lim_{x\to 0}(x+5)=5.$$

Example 3:

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}.$$

In order to evaluate the limit, we need to recall that the difference of squares of real numbers can be factored as $x^2-y^2=(x+y)(x-y)$.

We then rewrite and simplify the original function as follows:

$$\frac{\sqrt{x}-3}{x-9}=\frac{\sqrt{x}-3}{(\sqrt{x}+3)(\sqrt{x}-3)}=\frac{1}{(\sqrt{x}+3)}.$$

Hence $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}=\lim_{x\to 9}\frac{1}{\sqrt{x}+3}=\frac{1}{6}$.

You will solve similar examples in the homework where some clever applications of factoring to reduce fractions will enable you to evaluate the limit.

Limits of Composite Functions

While we can use the Properties to find limits of composite functions, composite functions will present some difficulties that we will fully discuss in the next Lesson. We can illustrate with the following examples, one where the limit exists and the other where the limit does not exist.

Example 4:

Consider $f(x)=\frac{1}{x+1}, g(x)=x^2$. Find $\lim_{x\to -1}(f\circ g)(x)$.

We see that $(f\circ g)(x)=\frac{1}{x^2+1}$ and note that property #5 does hold. Hence by direct substitution we have $\lim_{x\to -1}(f\circ g)(x)=\frac{1}{(-1)^2+1}=\frac{1}{2}$.

Example 5:

Consider $f(x)=\frac{1}{x+1}, g(x)=-1$. Then we have that $f(g(x))$ is undefined and we get the indeterminate form $1/0$. Hence $\lim_{x\to -1}(f\circ g)(x)$ does not exist.

Limits of Trigonometric Functions

In evaluating limits of trigonometric functions we will look to rely more on numerical and graphical techniques due to the unique behavior of these functions. Let’s look at a couple of examples.

Example 6:

Find $\lim_{x\to 0}\sin(x)$.

We can find this limit by observing the graph of the sine function and using the **[CALC VALUE]** function of our calculator to show that $\lim_{x \to 0} \sin x = 0$.

While we could have found the limit by direct substitution, in general, when dealing with trigonometric functions, we will rely less on formal properties of limits for finding limits of trigonometric functions and more on our graphing and numerical techniques.

The following theorem provides us a way to evaluate limits of complex trigonometric expressions.

Squeeze Theorem

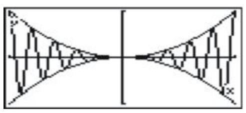
Suppose that $f(x) \le g(x) \le h(x)$ for x near a, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.

Then $\lim_{x \to a} g(x) = L$.

In other words, if we can find bounds for a function that have the same limit, then the limit of the function that they bound must have the same limit.

Example 7:

Find $\lim_{x \to 0} x^2 \cos(10\pi x)$.



From the graph we note that:

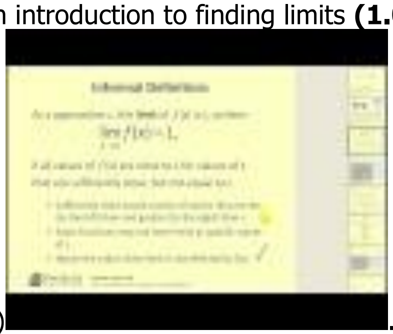
1. The function is bounded by the graphs of x^2 and $-x^2$
2. $\lim_{x \to 0} x^2 = \lim_{x \to 0}(-x^2) = 0$.

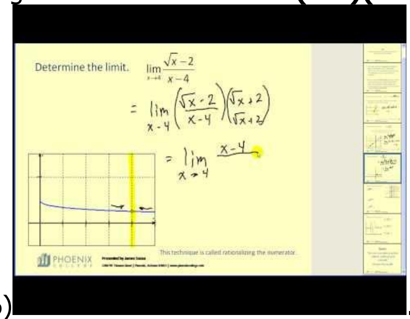
Hence the Squeeze Theorem applies and we conclude that $\lim_{x \to 0} x^2 \cos(10\pi x) = 0$.

Lesson Summary

1. We learned to find the limit of basic functions.
2. We learned to find the limit of polynomial, rational and radical functions.
3. We learned how to find limits of composite and trigonometric functions.
4. We used the Squeeze Theorem to find special limits.

Multimedia Links

For an introduction to finding limits **(1.0)**, see Math Video Tutorials by James Sousa, Introduction to Limits (8:46).

For more examples finding the limits of functions **(1.1)(1.2)**, see [Math Video Tutorials by James Sousa, Determining the Limits of Functions]() (8:46)

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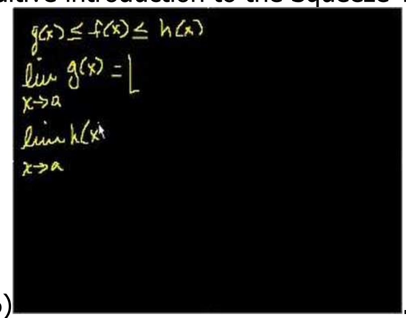
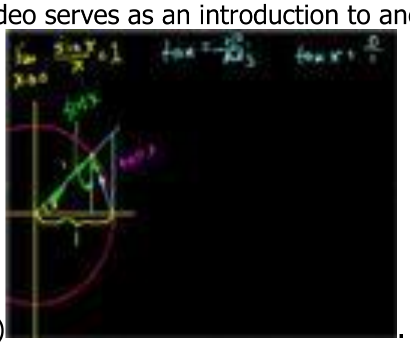
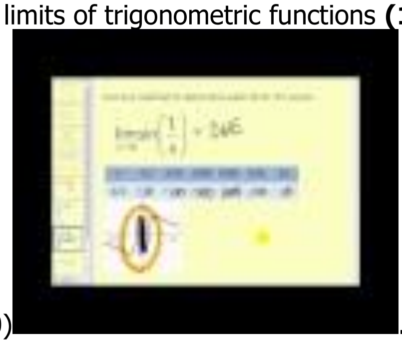
For a brief, intuitive introduction to the Squeeze Theorem using everyday examples **(1.1)**, see [Khan Academy Squeeze Theorem]() (7:36). This video includes a graphical presentation of the Squeeze Theorem. This video serves as an introduction to another (much longer) video, [Khan Academy Proof of Limit x -> 0 sin(x)/x]() (18:04).

For a video on determining limits of trigonometric functions **(1.2)(1.3)**, see [Math Video Tutorials by James Sousa, Introduction to Limits]() (7:20).

Review Questions

Find each of the following limits if they exist.

1. $\lim_{x\to 2}(x^2-3x+4)$
2. $\lim_{x\to 4}\frac{x^2-16}{x-4}$
3. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$
4. $\lim_{x\to -1}\frac{x-2}{x+1}$
5. $\lim_{x\to -1}\frac{10x-2}{3x+1}$

6. $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$
7. $\lim_{x\to 5} \frac{x^2-25}{x^3-125}$
8. Consider $f(x) = \frac{1}{x+1}, g(x) = x^2$. We found $\lim_{x\to -1}(f\circ g)(x) = \frac{1}{2}$. Find $\lim_{x\to -1}(g\circ f)(x)$.
9. Consider function $f(x)$ such that $5x - 11 \le f(x) \le x^2 - 4x + 9$ for $x \ge 0$. Use the Squeeze Theorem to find $\lim_{x\to 5} f(x)$.
10. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^4 \sin(\frac{1}{x}) = 0$

Review Answers

1. $\lim_{x\to 2}(x^2 - 3x + 4) = 2$
2. $\lim_{x\to 4} \frac{x^2-16}{x-4} = 8$
3. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \frac{1}{4}$
4. $\lim_{x\to -1} \frac{x-2}{x+1}$ does not exist.
5. $\lim_{x\to -1} \frac{10x-2}{3x+1} = 6$
6. $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \frac{1}{4}$
7. $\lim_{x\to 5} \frac{x^2-25}{x^3-125} = \frac{2}{15}$
8. $\lim_{x\to -1}(g\circ f)(x)$ does not exist since $g(f(x))$ is undefined.
9. $\lim_{x\to 5} f(x) = 14$ since $\lim_{x\to 5}(5x-11) = \lim_{x\to 5}(x^2 - 4x + 9) = 14$
10. Note that $x^4 \ge \sin(\frac{1}{x}) \ge -x^4$, and since $\lim_{x\to 0} x^4 = \lim_{x\to 0}(-x^4) = 0$, then by the Squeeze Theorem we must have $\lim_{x\to 0} x^4 \sin(\frac{1}{x}) = 0$.

1.7 Continuity

Learning Objectives

A student will be able to:

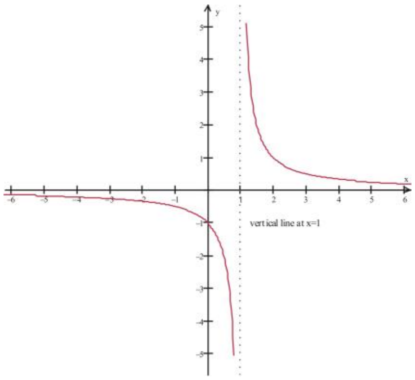
- Learn to examine continuity of functions.
- Find one-sided limits.
- Understand properties of continuous functions.
- Solve problems using the Min-Max theorem.
- Solve problems using the Intermediate Value Theorem.

Introduction

In this lesson we will discuss the property of continuity of functions and examine some very important implications. Let's start with an example of a rational function and observe its graph. Consider the following function:

$f(x) = (x+1)/(x^2-1).$

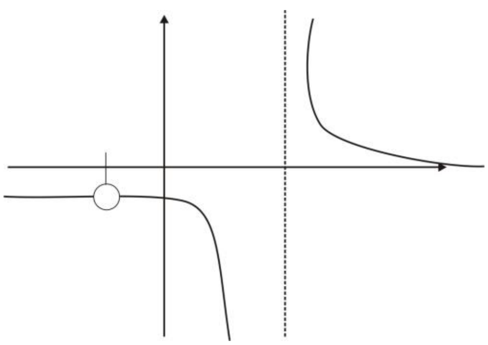
We know from our study of domains that in order for the function to be defined, we must use $x \neq -1, 1$. Yet when we generate the graph of the function (using the standard viewing window), we get the following picture that appears to be defined at $x = -1$:



The seeming contradiction is due to the fact that our original function had a common factor in the numerator and denominator, $x+1$, that cancelled out and gave us a picture that appears to be the graph of $f(x)=1/(x-1)$.

But what we actually have is the original function, $f(x)=(x+1)/(x^2-1)$, that we know is not defined at $x=-1$. At $x=-1$, we have a hole in the graph, or a discontinuity of the function at $x=-1$. That is, the function is defined for all other $x-$values close to $x=-1$.

Loosely speaking, if we were to hand-draw the graph, we would need to take our pencil off the page when we got to this hole, leaving a gap in the graph as indicated:



Now we will formalize the property of continuity of a function and provide a test for determining when we have continuous functions.

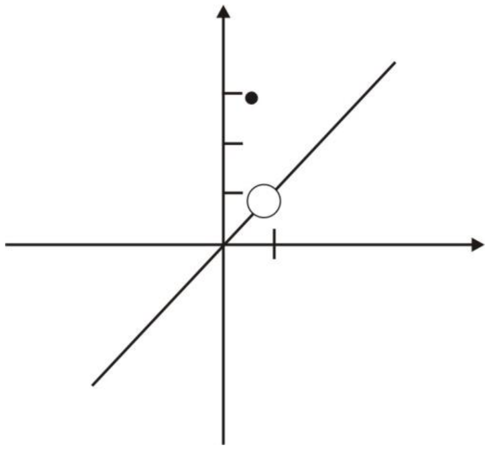
Continuity of a Function

Definition:

The function $f(x)$ is ***continuous at*** $x=a$ if the following conditions all hold:

1. a is in the domain of $f(x)$;
2. $\lim_{x\to a} f(x)$ exists;
3. $\lim_{x\to a} f(x)=f(a)$

Note that it is possible to have functions where two of these conditions are satisfied but the third is not. Consider the piecewise function

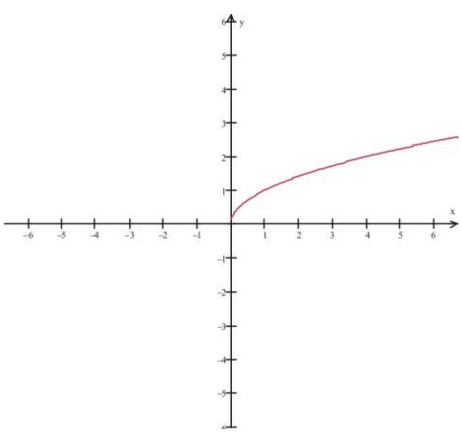


$$f(x) = \begin{cases} x, \text{ if } x \neq 1 \\ 3, \text{ if } x = 1 \end{cases}$$

In this example we have $\lim_{x\to 1} f(x)$exists, $x=1$is in the domain of $f(x)$, but $\lim_{x\to 1} f(x) \neq f(1)$.

One-Sided Limits and Closed Intervals

Let's recall our basic square root function, $f(x) = \sqrt{x}$.



Since the domain of $f(x) = \sqrt{x}$is $x \geq 0$, we see that that $\lim_{x\to 0} \sqrt{x}$does not exist. Specifically, we cannot find open intervals around $x=0$that satisfy the limit definition. However we do note that as we approach $x=0$from the right-hand side, we see the successive values tending towards $x=0$. This example provides some rationale for how we can define ***one-sided limits***.

Definition:

Similarly, we say that the ***left-hand limit*** of $f(x)$at $a$is b, written as $\lim_{x\to a^-} f(x) = b$, if for every open interval $N$of $b$there exists an open interval $(a-\delta, a)$contained in the domain of $f(x)$such that $f(x)$is in $N$for every $x$in $(a-\delta, a)$.

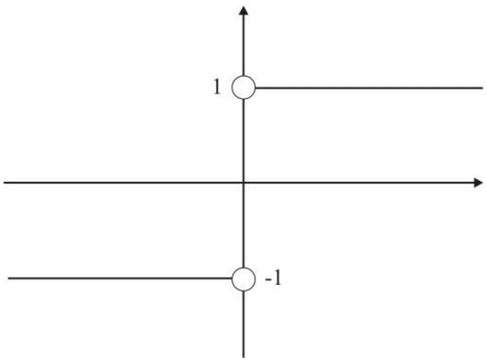
For the example above, we write $\lim_{x\to 0^+}\sqrt{x} = 0$.

We say that the ***right-hand*** *limit* of a function $f(x)$at $a$is b, written as $\lim_{x\to a^+} f(x) = b$, if for every open interval $N$of b, there exists an open interval $(a, a+\delta)$contained in the domain of $f(x)$such that $f(x)$is in N for every $x$in $(a, a+\delta)$.

Example 1:

Find $\lim_{x\to 0^+} \frac{x}{|x|}$.

The graph has a discontinuity at $x = 0$ as indicated:



We see that $\lim_{x\to 0^+} \frac{x}{|x|} = 1$ and also that $\lim_{x\to 0^-} \frac{x}{|x|} = -1$.

Properties of Continuous Functions

Let's recall our example of the limit of composite functions:

$f(x) = 1/(x+1),\ g(x) = -1.$

We saw that $f(g(x))$ is undefined and has the indeterminate form of $1/0$. Hence $\lim_{x\to -1}(f\circ g)(x)$ does not exist.

In general, we will require that f be continuous at $x = g(a)$ and $x = g(a)$ must be in the domain of $(f\circ g)$ in order for $\lim_{x\to a}(f\circ g)(x)$ to exist.

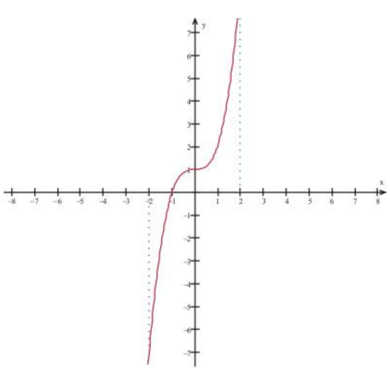
We will state the following theorem and delay its proof until Chapter 3 when we have learned more about real numbers.

Min-Max Theorem: If a function $f(x)$ is continuous in a closed interval I, then $f(x)$ has both a maximum value and a minimum value in I.

Example 2:

Consider $f(x) = x^3 + 1$ and interval $I = [-2, 2]$.

The function has a minimum value at value at $x = -2, f(-2) = -7,$ and a maximum value at $x = 2,$ where $f(2) = 9$



We will conclude this lesson with a theorem that will enable us to solve many practical problems such as finding zeros of functions and roots of equations.

Intermediate Value Theorem

If a function is continuous on a closed interval $[a, b]$, then the function assumes every value between $f(a)$ and $f(b)$.

The proof is left as an exercise with some hints provided. (Homework #10).

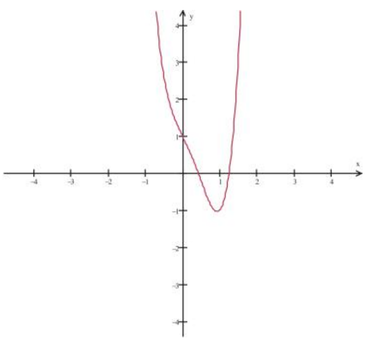
We can use the Intermediate Value Theorem to analyze and approximate zeros of functions.

Example 3:

Use the Intermediate Value Function to show that there is at least one zero of the function in the indicated interval.

$$f(x) = 3x^4 - 3x^3 - 2x + 1, \ (1, 2)$$

We recall that the graph of this function is shaped somewhat like a parabola; viewing the graph in the standard window, we get the following graph:



Of course we could zoom in on the graph to see that the lowest point on the graph lies within the fourth quadrant, but let's use the **[CALC VALUE]** function of the calculator to verify that there is a zero in the interval $(1, 2)$. In order to apply the Intermediate Value Theorem, we need to find a pair of $x-$values that have function values with different signs. Let's try some in the table below.

x	$f(x)$
1.1	$-.80$
1.2	$-.36$
1.3	$.37$

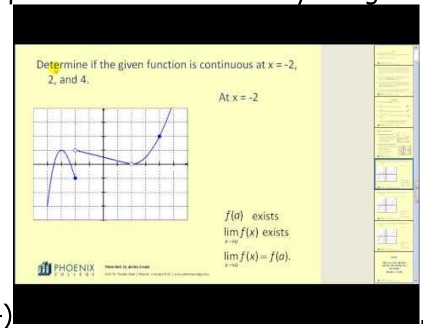
We see that the sign of the function values changes from negative to positive somewhere between 1.2 and 1.3. Hence, by the Intermediate Value theorem, there is some value c in the interval $(1.2, 1.3)$ such that $f(c) = 0$.

Lesson Summary

1. We learned to examine continuity of functions.
2. We learned to find one-sided limits.
3. We observed properties of continuous functions.
4. We solved problems using the Min-Max theorem.
5. We solved problems using the Intermediate Value Theorem.

Multimedia Links

For a presentation of continuity using limits **(2.0)**, see [Math Video Tutorials by James Sousa, Continuity Using Limits]

(5:44) 

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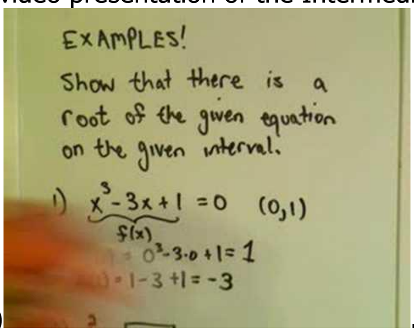
For a video presentation of the Intermediate Value Theorem **(3.0)**, see [Just Math Tutoring, Intermediate Value Theorem]

(7:53) 

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Review Questions

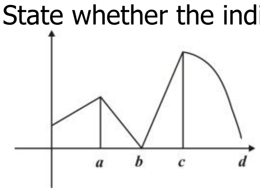
1. Generate the graph of $f(x) = (|x+1|)/(x+1)$ using your calculator and discuss the continuity of the function.
2. Generate the graph of $f(x) = (3x-6)/(x^2-4)$ using your calculator and discuss the continuity of the function.

Compute the limits in #3 - 6.

3. $\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{1+\sqrt{x}-1}}$
4. $\lim_{x \to 2^-} \frac{x^3-8}{|x-2|(x-2)}$
5. $\lim_{x \to 1^+} \frac{2x|x-1|}{x-1}$
6. $\lim\limits_{x \to -2^-} \frac{|x+2|+x+2}{|x+2|-x-2}$

In problems 7 and 8, explain how you know that the function has a root in the given interval. (Hint: Use the Intermediate Value Function to show that there is at least one zero of the function in the indicated interval.):

7. $f(x) = x^3 + 2x^2 - x + 1$, in the interval (-3, -2)
8. $f(x) = \sqrt{x} - \sqrt[3]{x} - 1$, in the interval (9, 10)
9. State whether the indicated x-values correspond to maximum or minimum values of the function depicted below.



10. Prove the Intermediate Value Theorem: If a function is continuous on a closed interval $[a, b]$, then the function assumes every value between $f(a)$ and $f(b)$.

Review Answers

1. While graph of the function appears to be continuous everywhere, a check of the table values indicates that the function is not continuous at x = -1.
2. While the function appears to be continuous for all $x \neq -2$, a check of the table values indicates that the function is not continuous at x = 2.
3. $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{1+\sqrt{x}-1}} = 2$
4. $\lim_{x\to 2^-} \frac{x^3-8}{|x-2|(x-2)}$ does not exist
5. $\lim_{x\to 1^+} \frac{2x|x-1|}{(x-1)} = 2$
6. $\lim_{x\to -2^-} \frac{|x+2|+x+2}{|x+2|-x-2} = 0$
7. $f(-2.5) = .375, f(-2.9) = -3.669.$ By the Intermediate Value Theorem, there is an x-value c with $f(c) = 0$.
8. $f(9.1) = -.071, f(9.99) = .006.$ By the Intermediate Value Theorem, there is an x-value c with $f(c) = 0$.
9. $x = a$ is a relative maximum, $x = b$ is an absolute minimum, $x = c$ is an absolute maximum and $x = d$ is not a maximum nor a minimum.
10. Here is an outline of the proof: we need to show that for every number d between f(a) and f(b), there exists a number c such that f(c) = d. Assume that f(a) < f(c) < f(b). Let S be the set of $x \in [a,b]$ for which f(x) < d. Note that $a \in S, b \in S$, so b is an upper bound for set S. Hence by the completeness property of the real numbers, S has an upper bound, c. There are then three possibilities to explore: f(c) < d, f(c) = d, or f(c) > d. Explore these and show why f(c) = d.

1.8 Infinite Limits

Learning Objectives

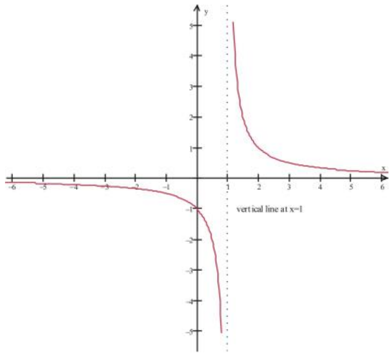
A student will be able to:

- Find infinite limits of functions.
- Analyze properties of infinite limits.
- Identify asymptotes of functions.
- Analyze end behavior of functions.

Introduction

In this lesson we will discuss infinite limits. In our discussion the notion of infinity is discussed in two contexts. First, we can discuss infinite limits in terms of the value a function as we increase x without bound. In this case we speak of the ***limit of*** $f(x)$ ***as*** x ***approaches*** ∞ and write $\lim_{x\to\infty} f(x)$. We could similarly refer to the ***limit of*** $f(x)$ ***as*** x ***approaches*** $-\infty$ and write $\lim_{x\to-\infty} f(x)$.

The second context in which we speak of infinite limits involves situations where the function values increase without bound. For example, in the case of a rational function such as $f(x) = \frac{x+1}{x^2-1}$, a function we discussed in previous lessons:



At $x = 1$, we have the situation where the graph grows without bound in both a positive and a negative direction. We say that we have a vertical asymptote at $x = 1$, and this is indicated by the dotted line in the graph above.

In this example we note that $\lim_{x\to 1} f(x)$does not exist. But we could compute both one-sided limits as follows.

$\lim_{x\to 1^-} f(x) = -\infty$and $\lim_{x\to 1^-} f(x) = +\infty$.

More formally, we define these as follows:

Definition:

We observe that as $x$increases in the positive direction, the function values tend to get smaller. The same is true if we decrease $x$in the negative direction. Some of these extreme values are indicated in the following table.
Suppose we look at the function $f(x) = (x + 1)/(x^2 - 1)$and determine the infinite limits $\lim_{x\to\infty} f(x)$and $\lim_{x\to-\infty} f(x)$.
The definition for negative infinite limits is similar.
The right-hand limit of the function $f(x)$at $x = a$is infinite, and we write $\lim_{x\to a^+} f(x) = \infty$, if for every positive number k, there exists an open interval $(a, a + \delta)$contained in the domain of $f(x)$, such that $f(x)$is in $(k, \infty)$for every $x$in $(a, a + \delta)$.

x	$f(x)$
100	$.0101$
200	$.0053$
-100	$-.0099$
-200	$-.005$

The following example shows how we can use this fact in evaluating limits of rational functions.
Since our original function was roughly of the form $f(x) = \frac{1}{x}$, this enables us to determine limits for all other functions of the form $f(x) = \frac{1}{x^p}$with $p > 0$.Specifically, we are able to conclude that $\lim_{x\to\infty} \frac{1}{x^p} = 0$. This shows how we can find infinite limits of functions by examining the ***end behavior*** of the function $f(x) = \frac{1}{x^p},\ p > 0.$
We observe that the values are getting closer to $f(x) = 0$.Hence $\lim_{x\to\infty} f(x) = 0$and $\lim_{x\to-\infty} f(x) = 0$

Example 1:

Find $\lim_{x\to\infty} \frac{2x^3 - x^2 + x - 1}{x^6 - x^5 + 3x^4 - 2x + 1}$.

Solution:

Note that we have the indeterminate form, so Limit Property #5 does not hold. However, if we first divide both numerator and denominator by the quantity x^6, we will then have a function of the form

$$\frac{f(x)}{g(x)} = \frac{\frac{2x^3}{x^6} - \frac{x^2}{x^6} + \frac{x}{x^6} - \frac{1}{x^6}}{\frac{x^6}{x^6} - \frac{x^5}{x^6} + \frac{3x^4}{x^6} - \frac{2x}{x^6} + \frac{1}{x^6}} = \frac{\frac{2}{x^3} - \frac{1}{x^4} + \frac{1}{x^5} - \frac{1}{x^6}}{1 - \frac{1}{x} + \frac{3}{x^2} - \frac{2}{x^5} + \frac{1}{x^6}}.$$

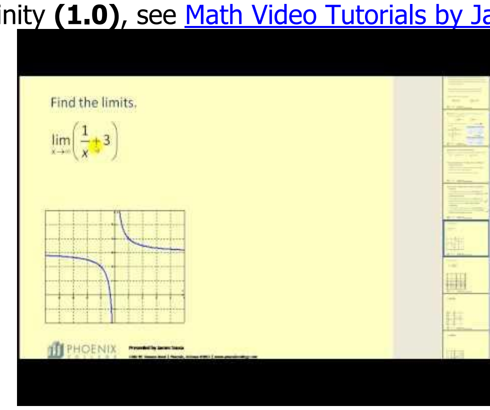
We observe that the limits $\lim_{x\to\infty} f(x)$and $\lim_{x\to\infty} g(x)$both exist. In particular, $\lim_{x\to\infty} f(x) = 0$and $\lim_{x\to\infty} g(x) = 1$. Hence Property #5 now applies and we have $\lim_{x\to\infty} \frac{2x^3-x^2+x-1}{x^6-x^5+3x^4-2x+1} = \frac{0}{1} = 0$.

Lesson Summary

1. We learned to find infinite limits of functions.
2. We analyzed properties of infinite limits.
3. We identified asymptotes of functions.
4. We analyzed end behavior of functions.

Multimedia Links

For more examples of limits at infinity **(1.0)**, see Math Video Tutorials by James Sousa, Limits at Infinity (9:42)



Review Questions

In problems 1 - 7, find the limits if they exist.

1. $\lim_{x\to 3^+} \frac{(x+2)^2}{(x-2)^2-1}$
2. $\lim_{x\to\infty} \frac{(x+2)^2}{(x-2)^2-1}$
3. $\lim_{x\to 1^+} \frac{(x+2)^2}{(x-2)^2-1}$
4. $\lim_{x\to\infty} \frac{2x-1}{x+1}$
5. $\lim_{x\to-\infty} \frac{x^5+3x^4+1}{x^3-1}$
6. $\lim_{x\to\infty} \frac{3x^4-2x^2+3x+1}{2x^4-2x^2+x-3}$
7. $\lim_{x\to\infty} \frac{2x^2-x+3}{x^5-2x^3+2x-3}$

In problems 8 - 10, analyze the given function and identify all asymptotes and the end behavior of the graph.

8. $f(x) = \frac{(x+4)^2}{(x-4)^2-1}$
9. $f(x) = -3x^3 - x^2 + 2x + 2$
10. $f(x) = \frac{2x^2-8}{x+2}$

Review Answers

1. $\lim_{x\to 3^+} \frac{(x+2)^2}{(x-2)^2-1} = +\infty$
2. $\lim_{x\to\infty} \frac{(x+2)^2}{(x-2)^2-1} = 1$
3. $\lim_{x\to 1^+} \frac{(x+2)^2}{(x-2)^2-1} = -\infty$
4. $\lim_{x\to\infty} \frac{2x-1}{x+1} = 2$
5. ∞
6. $\lim_{x\to\infty} \frac{3x^4-2x^2+3x+1}{2x^4-2x^2+x-3} = \frac{3}{2}$
7. $\lim_{x\to\infty} \frac{2x^2-x+3}{x^5-2x^3+2x-3} = 0$
8. Zero at $x = -4$; vertical asymptotes at $x = 3, 5$; $f(x) \to 1$ as $x \to \pm\infty$.
9. Zero at $x = 1$; no vertical asymptotes; $f(x) \to -\infty$ as $x \to \infty$; $f(x) \to \infty$ as $x \to -\infty$.
10. Zero at $x = 2$; no vertical asymptotes but there is a discontinuity at $x = -2$; $f(x) \to -\infty$ as $x \to -\infty$; $f(x) \to \infty$ as $x \to \infty$.