2.1 Tangent Lines and Rates of Change

Learning Objectives

A student will be able to:

- Demonstrate an understanding of the slope of the tangent line to the graph.
- Demonstrate an understanding of the instantaneous rate of change.

A car speeding down the street, the inflation of currency, the number of bacteria in a culture, and the AC voltage of an electric signal are all examples of quantities that change with time. In this section, we will study the rate of change of a quantity and how it is related to the tangent lines on a curve.

The Tangent Line

If two points \(P(x_0, y_0)\) and \(Q(x_1, y_1)\) are two different points on the curve \(y = f(x)\) (Figure 1), then the slope of the secant line connecting the two points is given by

\[
m_{\text{sec}} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (1)
\]

Now if we let the point \(x_1\) approach \(x_0\), \(Q\) will approach \(P\) along the graph of \(f(x)\). Thus the slope of the secant line will gradually approach the slope of the tangent line as \(x_0\) approaches \(x_1\). Therefore (1) becomes

\[
m_{\text{tan}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (2)
\]

If we let \(h = x_1 - x_0\) then \(x_1 = x_0 + h\) and \(h \to 0\) becomes equivalent to \(x_1 \to x_0\), so (2) becomes

\[
m_{\text{tan}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}
\]

If the point \(P(x_0, y_0)\) is on the curve \(f\), then the tangent line at \(P\) has a slope that is given by

\[
m_{\text{tan}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}
\]

provided that the limit exists.

Recall from algebra that the point-slope form for the tangent line is given by

\[
y - y_0 = m_{\text{tan}}(x - x_0).
\]

Example 1:

Find the slope of the tangent line to the curve \(f(x) = x^3\) passing through point \(P(2, 8)\).

Solution:

Since \(P(x_0, y_0) = (2, 8)\), using the slope of the tangent equation,
\[ m_{\text{tan}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \]

we get

\[ m_{\text{tan}} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} \]
\[ = \lim_{h \to 0} \frac{(h^3 + 6h^2 + 12h + 8) - 8}{h} \]
\[ = \lim_{h \to 0} \frac{h^3 + 6h^2 + 12h}{h} \]
\[ = \lim_{h \to 0} (h^2 + 6h + 12) \]
\[ = 12. \]

Thus the slope of the tangent line is 12. Using the point-slope formula above,

\[ y - 8 = 12(x - 2) \]

or

\[ y = 12x - 16 \]

Next we are interested in finding a formula for the slope of the tangent line at any point on the curve \( f \). Such a formula would be the same formula that we are using except we replace the constant \( x_0 \) by the variable \( x \). This yields

\[ m_{\text{tan}} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

We denote this formula by

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \]

where \( f'(x) \) is read "\( f \) prime of \( x \)." The next example illustrate its usefulness.

**Example 2:**

If \( f(x) = x^2 - 3 \) find \( f'(x) \) and use the result to find the slope of the tangent line at \( x = 2 \)and \( x = -1 \).

**Solution:**

Since

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \]

then
To find the slope, we simply substitute $x = 2$ into the result \( f'(x) \),

\[
f'(x) = 2x
\]

\[
f'(2) = 2(2) = 4
\]

and

\[
f'(x) = 2x
\]

\[
f'(-1) = 2(-1) = -2
\]

Thus slopes of the tangent lines at $x = 2$ and $x = -1$ are $4$ and $-2$, respectively.

**Example 3:**

Find the slope of the tangent line to the curve $y = \frac{1}{x}$ that passes through the point $(1, 1)$.

**Solution:**

Using the slope of the tangent formula

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},
\]

and substituting $y = \frac{1}{x}$,

\[
y' = \lim_{h \to 0} \frac{\left(\frac{1}{x+h}\right) - \frac{1}{x}}{h}
\]

\[
= \lim_{h \to 0} \frac{x - x - h}{x(x + h)h}
\]

\[
= \lim_{h \to 0} \frac{-h}{hx(x + h)}
\]

\[
= \lim_{h \to 0} \frac{-1}{x(x + h)}
\]

\[
= \lim_{h \to 0} \frac{-1}{x(x + h)}
\]

\[
= \frac{-1}{x^2}.
\]

Substituting $x = 1$,
Thus the slope of the tangent line at \( x = 1 \) for the curve \( y = \frac{1}{x} \) is \( m = -1 \). To find the equation of the tangent line, we simply use the point-slope formula,

\[
y - y_0 = m(x - x_0),
\]

where \((x_0, y_0) = (1, 1)\).

\[
y - 1 = -1(x - 1)
\]

\[
= -x + 1 + 1
\]

\[
= -x + 2,
\]

which is the equation of the tangent line.

**Average Rates of Change**

The primary concept of calculus involves calculating the rate of change of a quantity with respect to another. For example, speed is defined as the rate of change of the distance travelled with respect to time. If a person travels 120 miles in four hours, his speed is \( \frac{120}{4} = 30 \text{ mi/hr} \). This speed is called the average speed or the average rate of change of distance with respect to time. Of course the person who travels 120 miles at a rate of 30 mi/hr for four hours does not do so continuously. He must have slowed down or sped up during the four-hour period. But it does suffice to say that he traveled for four hours at an average rate of 30 miles per hour. However, if the driver strikes a tree, it would not be his average speed that determines his survival but his speed at the instant of the collision. Similarly, when a bullet strikes a target, it is not the average speed that is significant but its instantaneous speed at the moment it strikes. So here we have two distinct kinds of speeds, average speed and instantaneous speed.

The average speed of an object is defined as the object’s displacement \( \Delta x \) divided by the time interval \( \Delta t \) during which the displacement occurs:

\[
v = \frac{\Delta x}{\Delta t} = \frac{x_1 - x_0}{t_1 - t_0}.
\]

Notice that the points \((t_0, x_0)\) and \((t_1, x_1)\) lie on the position-versus-time curve, as Figure 1 shows. This expression is also the expression for the slope of a secant line connecting the two points. Thus we conclude that the average velocity of an object between time \( t_0 \) and \( t_1 \) is represented geometrically by the slope of the secant line connecting the two points \((t_0, x_0)\) and \((t_1, x_1)\). If we choose \( t_1 \) close to \( t_0 \), then the average velocity will closely approximate the instantaneous velocity at time \( t_0 \).
Geometrically, the average rate of change is represented by the slope of a secant line and the instantaneous rate of change is represented by the slope of the tangent line (Figures 2 and 3).

**Average Rate of Change** (such as the average velocity)

The average rate of change of \( x = f(t) \) over the time interval \([t_0, t_1]\) is the slope of the secant line to the points \((t_0, f(t_0))\) and \((t_1, f(t_1))\) on the graph (Figure 2).

\[
m_{\text{sec}} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}
\]

**Instantaneous Rate of Change**

The instantaneous rate of change of \( x = f(t) \) at the time \( t_0 \) is the slope of the tangent line at the time \( t_0 \) on the graph.

\[
m_{\text{tan}} = f'(t_0) = \lim_{t_1 \to t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0}
\]

**Example 4:**

Suppose that \( y = x^2 - 3 \).

1. Find the average rate of change of \( y \) with respect to \( x \) over the interval \([0, 2]\).
2. Find the instantaneous rate of change of \( y \) with respect to \( x \) at the point \( x = -1 \).
**Solution:**

1. Applying the formula for Average Rate of Change with $f(x) = x^2 - 3$ and $x_0 = 0$ and $x_1 = 2$ yields

$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(2) - f(0)}{2 - 0} = \frac{1 - (-3)}{2} = 2$$

This means that the average rate of change of $y$ is 2 units per unit increase in $x$ over the interval $[0, 2]$.

2. From the example above, we found that $f(x) = 2x$, so

$$m_{tan} = f'(x_0) = f'(-1) = 2(-1) = -2$$

This means that the instantaneous rate of change is negative. That is, $y$ is decreasing at $x = -1$. It is decreasing at a rate of 2 units per unit increase in $x$.

**Multimedia Links**

For a video explaining instantaneous rates of change (4.2), see *Slopes of Tangents and Instantaneous Rates of Change* (9:26).

For a video with an application regarding velocity (4.2), see *Calculus Help: Instantaneous Rates of Change* (9:03).

The following applet illustrates how the slope of a secant line can become the slope of the tangent to a curve at a point as $h \to 0$. Follow the directions on the page to explore how changing the distance between the two points makes the
slope of the secant approach the slope of the tangent Slope at a Point Applet. Note that the function and the point of tangency can also be edited in this simulator.

**Review Questions**

1. Given the function \( y = \frac{1}{2} x^2 \) and the values of \( x_0 = 3 \) and \( x_1 = 4 \), find:
   a. The average rate of change of \( y \) with respect to \( x \) over the interval \([x_0, x_1]\).
   b. The instantaneous rate of change of \( y \) with respect to \( x \) at \( x_0 \).
   c. The slope of the tangent line at \( x_1 \).
   d. The slope of the secant line between points \( x_0 \) and \( x_1 \).
   e. Make a sketch of \( y = \frac{1}{2} x^2 \) and show the secant and tangent lines at their respective points.

2. Repeat problem #1 for \( f(x) = \frac{1}{x} \) and the values \( x_0 = 2 \) and \( x_1 = 3 \).

3. Find the slope of the graph \( f(x) = x^2 + 1 \) at a general point \( x \). What is the slope of the tangent line at \( x_0 = 6 \)?

4. Suppose that \( y = \frac{1}{\sqrt{x}} \).
   a. Find the average rate of change of \( y \) with respect to \( x \) over the interval \([1, 3]\).
   b. Find the instantaneous rate of change of \( y \) with respect to \( x \) at point \( x = 1 \).

5. A rocket is propelled upward and reaches a height of \( h(t) = 4.9t^2 \) in \( t \) seconds.
   a. How high does it reach in 35 seconds?
   b. What is the average velocity of the rocket during the first 35 seconds?
   c. What is the average velocity of the rocket during the first 200 meters?
   d. What is the instantaneous velocity of the rocket at the end of the 35 seconds?

6. A particle moves in the positive direction along a straight line so that after \( t \) nanoseconds, its traversed distance is given by \( x(t) = 9.9t^3 \) nanometers.
   a. What is the average velocity of the particle during the first 2 nanoseconds?
   b. What is the instantaneous velocity of the particle at \( t = 2 \) nanoseconds?

**Review Answers**

1.
   a. \( \frac{7}{3} \)
   b. \( \frac{3}{2} \)
   c. \( \frac{4}{3} \)
   d. \( \frac{2}{3} \)

2.
   a. \( -\frac{1}{6} \)
   b. \( \frac{3}{1} \)
   c. \( -\frac{9}{1} \)
   d. \( \frac{9}{6} \)

3. \( 2x, 12 \)

4.
   a. \( \left( \frac{\sqrt{3}}{6} - \frac{1}{2} \right) \)
   b. \( -\frac{1}{2} \)

5.
   a. 6002.5 m
   b. 171.5 m/s
   c. 31.3 m/s
   d. 343 m/s

6.
   a. 39.6 m/s   b. 118.8 m/s
2.2 The Derivative

Learning Objectives

A student will be able to:

- Demonstrate an understanding of the derivative of a function as a slope of the tangent line.
- Demonstrate an understanding of the derivative as an instantaneous rate of change.
- Understand the relationship between continuity and differentiability.

The function \( f'(x) \) that we defined in the previous section is so important that it has its own name.

The Derivative

The function \( f' \) is defined by the new function

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

where \( f \) is called the derivative of \( f \) with respect to \( x \). The domain of \( f \) consists of all the values of \( x \) for which the limit exists.

Based on the discussion in previous section, the derivative \( f' \) represents the slope of the tangent line at point \( x \). Another way of interpreting it is to say that the function \( y = f(x) \) has a derivative \( f' \) whose value at \( x \) is the instantaneous rate of change of \( y \) with respect to point \( x \).

Example 1:

Find the derivative of \( f(x) = \frac{x}{x+1} \).

Solution:

We begin with the definition of the derivative,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \left( \frac{1}{h} \right) \left[ f(x + h) - f(x) \right],
\]

where

\[
f(x) = \frac{x}{x+1} \quad \text{and} \quad f(x + h) = \frac{x + h}{x + h + 1}
\]

Substituting into the derivative formula,
\[ f'(x) = \lim_{h \to 0} \frac{1}{h} \left[ \frac{x + h}{x + h + 1} - \frac{x}{x + 1} \right] \]
\[ = \lim_{h \to 0} \frac{1}{h} \left[ \frac{(x + h)(x + 1) - x(x + h + 1)}{(x + h + 1)(x + 1)} \right] \]
\[ = \lim_{h \to 0} \frac{1}{h} \left[ \frac{x^2 + x + hx + h - xh - x}{(x + h + 1)(x + 1)} \right] \]
\[ = \lim_{h \to 0} \frac{1}{h} \left[ \frac{1}{(x + h + 1)(x + 1)} \right] \]
\[ = \lim_{h \to 0} \frac{1}{(x + h + 1)(x + 1)} \]
\[ = \frac{1}{(x + 1)^2}. \]

Example 2:

Find the derivative of \( f(x) = \sqrt{x} \) and the equation of the tangent line at \( x_0 = 1 \).

Solution:

Using the definition of the derivative,
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \]
\[ = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} \]
\[ = \lim_{h \to 0} \frac{1}{h} \frac{x + h - x}{\sqrt{x + h} + \sqrt{x}} \]
\[ = \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} \]
\[ = \frac{1}{2\sqrt{x}}. \]

Thus the slope of the tangent line at \( x_0 = 1 \) is
\[ f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}. \]

For \( x_0 = 1 \), we can find \( y_0 \) by simply substituting into \( f(x) \).
\[ f(x_0) \equiv y_0 \]
\[ f(1) = \sqrt{1} = 1 \]
\[ y_0 = 1. \]

Thus the equation of the tangent line is
\[ y - y_0 = m(x - x_0) \]
\[ y - 1 = \frac{1}{2}(x - 1) \]
\[ y = \frac{1}{2}x + \frac{1}{2}. \]

**Notation**

Calculus, just like all branches of mathematics, is rich with notation. There are many ways to denote the derivative of a function \( y = f(x) \) in addition to the most popular one, \( f'(x) \). They are:

\[
\begin{array}{cccc}
  f'(x) & \frac{dy}{dx} & y' & \frac{df}{dx} \\
  \frac{df(x)}{dx} & 
\end{array}
\]

In addition, when substituting the point \( x_0 \) into the derivative we denote the substitution by one of the following notations:

\[
\begin{array}{cccc}
  f'(x_0) & \frac{dy}{dx} \big|_{x = x_0} & \frac{df}{dx} \big|_{x = x_0} & \frac{df(x_0)}{dx} \\
\end{array}
\]

**Existence and Differentiability of a Function**

If, at the point \((x_0, f(x_0))\), the limit of the slope of the secant line does not exist, then the derivative of the function \( f(x) \) at this point does not exist either. That is,

\[
m_{sec} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{Does not exist}
\]

If \( m_{sec} \), then the derivative \( f'(x) \) also fails to exist as \( x \to x_0 \). The following examples show four cases where the derivative fails to exist.

1. **At a corner.** For example \( f(x) = |x| \), where the derivative on both sides of \( x = 0 \) differ (Figure 4).
2. **At a cusp.** For example \( f(x) = x^{2/3} \), where the slopes of the secant lines approach \(+\infty\) on the right and \(-\infty\) on the left (Figure 5).
3. **A vertical tangent.** For example \( f(x) = x^{1/3} \), where the slopes of the secant lines approach \(+\infty\) on the right and \(-\infty\) on the left (Figure 6).
4. **A jump discontinuity.** For example, the step function (Figure 7)

\[
f(x) = \begin{cases} 
-2, & x < 0 \\
2, & x \geq 0
\end{cases}
\]

where the limit from the left is \(-2\) and the limit from the right is \(2\).
Many functions in mathematics do not have corners, cusps, vertical tangents, or jump discontinuities. We call them **differentiable functions**.

From what we have learned already about differentiability, it will not be difficult to show that continuity is an important condition for differentiability. The following theorem is one of the most important theorems in calculus:

**Differentiability and Continuity**

If $f$ is differentiable at $x_0$, then $f$ is also continuous at $x_0$.

The logically equivalent statement is quite useful: If $f$ is *not* continuous at $x_0$, then $f$ is not differentiable at $x_0$.

(The converse is not necessarily true.)

We have already seen that the converse is not true in some cases. The function can have a cusp, a corner, or a vertical tangent and still be continuous, but it is not differentiable.

**Multimedia Links**
For an introduction to the derivative (4.0)(4.1), see Math Video Tutorials by James Sousa, Introduction to the Derivative.

The following simulator traces the instantaneous slope of a curve and graphs a qualitative form of derivative function on an axis below the curve Surfing the Derivative.

The following applet allows you to explore the relationship between a function and its derivative on a graph. Notice that as you move x along the curve, the slope of the tangent line to \( f(x) \) is the height of the derivative function, \( f'(x) \). Derivative Applet. This applet is customizable—after doing the steps outlined on the page, feel free to change the function definition and explore the derivative of many functions.

For a video presentation of differentiability and continuity (4.3), see Differentiability and Continuity.

Review Questions

In problems 1-6, use the definition of the derivative to find \( f'(x) \) and then find the equation of the tangent line at \( x = x_0 \).

1. \( f(x) = 6x^2; x_0 = 3 \)
2. \( f(x) = \sqrt{x + 2}; x_0 = 8 \)
3. \( f(x) = 3x^3 - 2; x_0 = -1 \)
4. \( f(x) = \frac{1}{x^2}; x_0 = -1 \)
5. \( f(x) = ax^2 - b, \) where a and b are constants; \( x_0 = b \)
6. \( f(x) = x^{1/3}; x_0 = 1 \)
7. Find \( y'(1) \) given that \( y = 5x^2 - 2. \)
8. Show that \( f(x) = \sqrt{x} \) is continuous at \( x = 0 \) but not differentiable at \( x = 0 \). Sketch the graph.
9. Show that \( f(x) = \begin{cases} x^2 + 1 & x \geq 1 \\ 2x & x > 1 \end{cases} \) is continuous and differentiable at \( x = 1 \). Sketch the graph of \( f(x) \).
10. Suppose that \( f \) is a differentiable function and has the property that

\[
f(x + y) = f(x) + f(y) + 3xy \quad \text{and} \quad \lim_{h \to 0} \frac{f(h)}{h} = 4.
\]

Find \( f(0) \) and \( f'(x) \).
Review Answers

1. \( f'(x) = 12x \cdot y = 36x - 54 \)
2. \( f'(x) = \frac{1}{\sqrt{x^2+2}}, \quad y = \frac{1}{\sqrt{6}} \left( \frac{1}{2}x + 6 \right) \)
3. \( f'(x) = 9x^2 \cdot y = 9x + 4 \)
4. \( f'(x) = \frac{-1}{(x+2)^2}, \quad y = -x \)
5. \( f'(x) = 2ax, \quad y = 2abx - b(ab + 1) \)
6. \( f'(x) = \frac{1}{3x^{2/3}}, \quad y = \frac{1}{3} x + \frac{2}{3} \)
7. 10
8. Hint: Take the limit from both sides.
9. Hint: Take the limit from both sides.
10. \( f(0) = 0, \quad f'(x) = 4 + 3x \)