7.1 Indefinite Integrals Calculus

Learning Objectives

A student will be able to:

- Find antiderivatives of functions.
- Represent antiderivatives.
- Interpret the constant of integration graphically.
- Solve differential equations.
- Use basic antidifferentiation techniques.
- Use basic integration rules.

Introduction

In this lesson we will introduce the idea of the ***antiderivative*** of a function and formalize as ***indefinite integrals.*** We will derive a set of rules that will aid our computations as we solve problems.

Antiderivatives

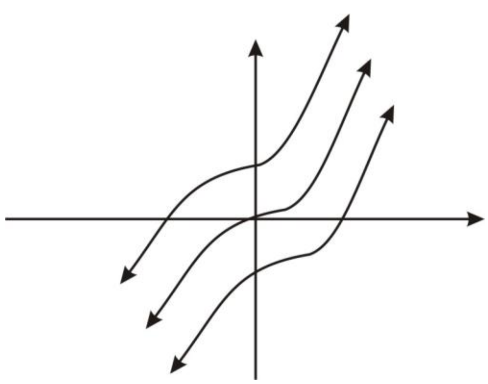
Definition

A function $F(x)$ is called an ***antiderivative*** of a function f if $F'(x) = f(x)$ for all x in the domain of f.

Example 1:

Consider the function $f(x) = 3x^2$. Can you think of a function $F(x)$ such that $F'(x) = f(x)$? ***(Answer:*** $F(x) = x^3$, $F(x) = x^3 - 6$, ***many other examples.)***

Since we differentiate $F(x)$ to get $f(x)$, we see that $F(x) = x^3 + C$ will work for any constant C. Graphically, we can think the set of all antiderivatives as vertical transformations of the graph of $F(x) = x^3$. The figure shows two such transformations.



With our definition and initial example, we now look to formalize the definition and develop some useful rules for computational purposes, and begin to see some applications.

Notation and Introduction to Indefinite Integrals

The process of finding antiderivatives is called ***antidifferentiation***, more commonly referred to as ***integration.*** We have a particular sign and set of symbols we use to indicate integration:

$$\int f(x)dx = F(x) + C.$$

We refer to the left side of the equation as "the indefinite integral of $f(x)$ with respect to x." The function $f(x)$ is called the ***integrand*** and the constant C is called the ***constant of integration.*** Finally the symbol dx indicates that we are to integrate with respect to x.

Using this notation, we would summarize the last example as follows:

$$\int 3x^2 dx = x^3 + C$$

Using Derivatives to Derive Basic Rules of Integration

As with differentiation, there are several useful rules that we can derive to aid our computations as we solve problems. The first of these is a rule for integrating power functions, $f(x) = x^n \; [n \neq -1]$, and is stated as follows:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C.$$

We can easily prove this rule. Let $F(x) = \frac{1}{n+1}x^{n+1} + C, n \neq -1$. We differentiate with respect to x and we have:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) + \frac{d}{dx}(C) \\ &= \left(\frac{1}{n+1}\right)\frac{d}{dx}\left(x^{n+1}\right) + \frac{d}{dx}(C) \\ &= \left(\frac{n+1}{n+1}\right)x^n + 0 \\ &= x^n. \end{aligned}$$

The rule holds for $f(x) = x^n[n \neq -1]$. What happens in the case where we have a power function to integrate with $n = -1$, say $\int x^{-1} dx = \int \frac{1}{x} dx$. We can see that the rule does not work since it would result in division by 0. However, if we pose the problem as finding $F(x)$ such that $F'(x) = \frac{1}{x}$, we recall that the derivative of logarithm functions had this form. In particular, $\frac{d}{dx}\ln x = \frac{1}{x}$. Hence

$$\int \frac{1}{x}dx = \ln x + C.$$

In addition to logarithm functions, we recall that the basic exponentional function, $f(x) = e^x$, was special in that its derivative was equal to itself. Hence we have

$$\int e^x dx = e^x + C.$$

Again we could easily prove this result by differentiating the right side of the equation above. The actual proof is left as an exercise to the student.

As with differentiation, we can develop several rules for dealing with a finite number of integrable functions. They are stated as follows:

If f and g are integrable functions, and C is a constant, then

$$\begin{aligned}\int [f(x)+g(x)]\,dx &= \int f(x)dx + \int g(x)dx,\\ \int [f(x)-g(x)]dx &= \int f(x)dx - \int g(x)dx,\\ \int [Cf(x)]dx &= C\int f(x)dx.\end{aligned}$$

Example 2:

Compute the following indefinite integral.

$$\int \left[2x^3 + \frac{3}{x^2} - \frac{1}{x}\right] dx.$$

Solution:

Using our rules we have

$$\begin{aligned}\int \left[2x^3 + \frac{3}{x^2} - -\frac{1}{x}\right] dx &= 2\int x^3 dx + 3\int \frac{1}{x^2}dx - \int \frac{1}{x}dx\\ &= 2\left(\frac{x^4}{4}\right) + 3\left(\frac{x^{-1}}{-1}\right) - \ln x + C\\ &= \frac{x^4}{2} - \frac{3}{x} - \ln x + C.\end{aligned}$$

Sometimes our rules need to be modified slightly due to operations with constants as is the case in the following example.

Example 3:

Compute the following indefinite integral:

$$\int e^{3x} dx.$$

Solution:

We first note that our rule for integrating exponential functions does not work here since $\frac{d}{dx}e^{3x} = 3e^{3x}$. However, if we remember to divide the original function by the constant then we get the correct antiderivative and have

$$\int e^{3x} dx = \frac{e^{3x}}{3} + C.$$

We can now re-state the rule in a more general form as

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C.$$

Differential Equations

We conclude this lesson with some observations about integration of functions. First, recall that the integration process allows us to start with function f from which we find another function $F(x)$ such that $F'(x) = f(x)$. This latter equation is called a ***differential equation.*** This characterization of the basic situation for which integration applies gives rise to a set of equations that will be the focus of the Lesson on The Initial Value Problem.

Example 4:

Solve the general differential equation $f'(x) = x^{\frac{2}{3}} + \sqrt{x}.$

Solution:

We solve the equation by integrating the right side of the equation and have

$$f(x) = \int f'(x)dx = \int x^{\frac{2}{3}}dx + \int \sqrt{x}dx.$$

We can integrate both terms using the power rule, first noting that $\sqrt{x} = x^{\frac{1}{2}}$, and have

$$f(x) = \int x^{\frac{2}{3}}dx + \int x^{\frac{1}{2}}dx = \frac{3}{5}x^{\frac{5}{3}} + \frac{2}{3}x^{\frac{3}{2}} + C.$$

Lesson Summary

1. We learned to find antiderivatives of functions.
2. We learned to represent antiderivatives.
3. We interpreted constant of integration graphically.
4. We solved general differential equations.
5. We used basic antidifferentiation techniques to find integration rules.
6. We used basic integration rules to solve problems.

Multimedia Link

The following applet shows a graph, $f(x)$ and its derivative, $f'(x)$. This is similar to other applets we've explored with a function and its derivative graphed side-by-side, but this time $f(x)$ is on the right, and $f'(x)$ is on the left. If you edit the definition of $f'(x)$, you will see the graph of $f(x)$ change as well. The c parameter adds a constant to $f(x)$. Notice that you can change the value of c without affecting $f'(x)$. Why is this? Antiderivative Applet.

Review Questions

In problems #1–3, find an antiderivative of the function

1. $f(x) = 1 - 3x^2 - 6x$
2. $f(x) = x - x^{\frac{2}{3}}$
3. $f(x) = \sqrt[5]{2x + 1}$

In #4–7, find the indefinite integral

4. $\int (2 + \sqrt{5}) dx$
5. $\int 2(x - 3)^3 dx$
6. $\int (x^2 \cdot \sqrt[3]{x}) dx$
7. $\int x + \dfrac{1}{x\sqrt[4]{x}} dx$
8. Solve the differential equation $f'(x) = 4x^3 - 3x^2 + x - 3$.
9. Find the antiderivative $F(x)$ of the function $f(x) = 2e^{2x} + x - 2$ that satisfies $F(0) = 5$.
10. Evaluate the indefinite integral $\int |x| dx$. (Hint: Examine the graph of $f(x) = |x|$.)

Review Answers

1. $F(x) = x - x^3 - 3x^2 + C$
2. $F(x) = \frac{x^2}{2} - \frac{3}{5}x^{\frac{5}{3}} + C$
3. $F(x) = \frac{5}{12}(2x + 1)^{\frac{6}{5}} + C$
4. $\int (2 + \sqrt{5})\, dx = 2x + \sqrt{5}x + C$
5. $\int 2(x - 3)^3 dx = \frac{(x-3)^4}{2} + C$
6. $\int (x^2 \cdot \sqrt[3]{x})\, dx = \frac{3}{10}x^{\frac{10}{3}} + C$
7. $\int x + \dfrac{1}{x\sqrt[4]{x}} dx = \dfrac{x^2}{2} - \dfrac{4}{\sqrt[4]{x}} + C$
8. $f(x) = x^4 - x^3 + \frac{x^2}{2} - 3x + C$
9. $F(x) = e^{2x} + \frac{x^2}{x} - 2x + 4$
10. $\int |x| dx = \frac{x^2}{2} + C$

Indefinite Integrals Practice

1. Verify the statement by showing that the derivative of the right side is equal to the integrand of the left side.

   a. $\int\left(-\frac{9}{x^4}\right)dx = \frac{3}{x^3} + C$ b. $\int\left(1 - \frac{1}{\sqrt[3]{x^2}}\right)dx = x - 3\sqrt[3]{x} + C$

2. Integrate.

   a. $\int 6dx$ b. $\int 3t^2 dt$ c. $\int 5x^{-3}dx$ d. $\int du$

   e. $\int x^{3/2}dx$ f. $\int \sqrt[3]{x}dx$ g. $\int \frac{1}{x\sqrt{x}}dx$ h. $\int \frac{1}{2x^3}dx$

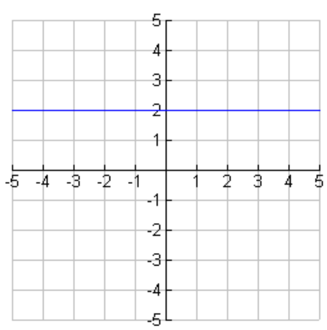
   i. $\int\left(x^3+2\right)dx$ j. $\int\left(2x^{4/3}+3x-1\right)dx$ k. $\int \sqrt[3]{x^2}dx$ l. $\int \frac{1}{x^3}dx$

   m. $\int \frac{1}{4x^2}dx$ n. $\int \frac{t^2+2}{t^2}dt$ o. $\int u\left(3u^2+1\right)du$

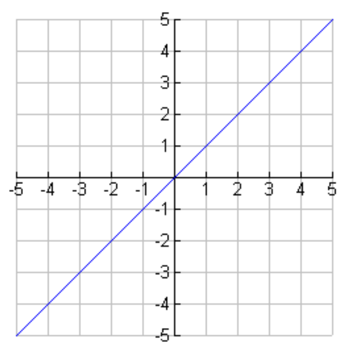
   p. $\int (x-1)(6x-5)dx$ q. $\int y^2\sqrt{y}dy$

3. Find two functions that have the given derivative and sketch the graph of each.

a.



b.



Answers: (Of course, you could have checked all of yours using differentiation!)

2. a. $6x + C$ b. $t^3 + C$ c. $-\frac{5}{2x^2} + C$ d. $u + C$

   e. $\frac{2}{5}x^{5/2} + C$ f. $\frac{3}{4}\sqrt[3]{x^4} + C$ g. $\frac{-2}{\sqrt{x}} + C$ h. $-\frac{1}{4x^2} + C$

   i. $\frac{x^4}{4} + 2x + C$ j. $\frac{6}{7}x^{7/3} + \frac{3}{2}x^2 - x + C$ k. $\frac{3}{5}x^{5/3} + C$ l. $-\frac{1}{2x^2} + C$ m. $-\frac{1}{4x} + C$

   n. $t - \frac{2}{t} + C$ o. $\frac{3}{4}u^4 + \frac{1}{2}u^2 + C$

   p. $2x^3 - \frac{11}{2}x^2 + 5x + C$ q. $\frac{2}{7}y^{7/2} + C$

7.2 The Initial Value Problem

Learning Objectives

- Find general solutions of differential equations
- Use initial conditions to find particular solutions of differential equations

Introduction

In the Lesson on Indefinite Integrals Calculus we discussed how finding antiderivatives can be thought of as finding solutions to differential equations: $F'(x) = f(x)$. We now look to extend this discussion by looking at how we can designate and find particular solutions to differential equations.

Let's recall that a general differential equation will have an infinite number of solutions. We will look at one such equation and see how we can impose conditions that will specify exactly one particular solution.

Example 1:

Suppose we wish to solve the following equation:

$$f'(x) = e^{3x} - 6\sqrt{x}.$$

Solution:

We can solve the equation by integration and we have

$$f(x) = \frac{1}{3}e^{3x} - 4x^{\frac{3}{2}} + C.$$

We note that there are an infinite number of solutions. In some applications, we would like to designate exactly one solution. In order to do so, we need to impose a condition on the function f. We can do this by specifying the value of f for a particular value of x. In this problem, suppose that add the condition that $f(0) = 1$. This will specify exactly one value of C and thus one particular solution of the original equation:

Substituting $f(0) = 1$ into our general solution $f(x) = \frac{1}{3}e^{3x} - 4x^{\frac{3}{2}} + C$ gives $1 = \frac{1}{3}e^{3(0)} - 4(0)^{\frac{3}{2}} + C$ or $C = 1 - \frac{1}{3} = \frac{2}{3}$. Hence the solution $f(x) = \frac{1}{3}e^{3x} - 4x^{\frac{3}{2}} + \frac{2}{3}$ is the ***particular solution*** of the original equation $f'(x) = e^{3x} - 6\sqrt{x}$ satisfying the ***initial condition*** $f(0) = 1.$

We now can think of other problems that can be stated as differential equations with initial conditions. Consider the following example.

Example 2:

Suppose the graph of f includes the point $(2, 6)$ and that the slope of the tangent line to f at any point x is given by the expression $3x + 4$. Find $f(-2)$.

Solution:

We can re-state the problem in terms of a differential equation that satisfies an initial condition.

$f'(x) = 3x + 4$ with $f(2) = 6$.

By integrating the right side of the differential equation we have

$f(x) = \frac{3}{2}x^2 + 4x + C$ as the general solution. Substituting the condition that $f(2) = 6$ gives

$$\begin{aligned} 6 &= \frac{3}{2}(2)^2 + 4(2) + C, \\ 6 &= 6 + 8 + C, \\ C &= -8. \end{aligned}$$

Hence $f(x) = \frac{3}{2}x^2 + 4x - 8$ is the ***particular solution*** of the original equation $f'(x) = 3x + 4$ satisfying the ***initial condition*** $f(2) = 6$.

Finally, since we are interested in the value $f(-2)$, we put -2 into our expression for f and obtain:

$f(-2) = -10$

Lesson Summary

1. We found general solutions of differential equations.
2. We used initial conditions to find particular solutions of differential equations.

Multimedia Link

The following applet allows you to set the initial equation for $f'(x)$ and then the slope field for that equation is displayed. In magenta you'll see one possible solution for $f(x)$. If you move the magenta point to the initial value, then you will see the graph of the solution to the initial value problem. Follow the directions on the page with the applet to explore this idea, and then try redoing the examples from this section on the applet. Slope Fields Applet.

Review Questions

In problems #1–3, solve the differential equation for $f(x)$.

1. $f'(x) = 2e^{2x} - 2\sqrt{x}$
2. $f'(x) = \sin x - \frac{1}{e^x}$
3. $f''(x) = (2 + x)\sqrt{x}$

In problems #4–7, solve the differential equation for $f(x)$ given the initial condition.

4. $f'(x) = 6x^5 - 4x^2 + \frac{7}{3}$ and $f(1) = 4$.
5. $f'(x) = 3x^2 + e^{2x}$ and $f(0) = 3$.
6. $f'(x) = \sqrt[3]{x^2} - \frac{1}{x^2}$ and $f(1) = 3$
7. $f'(x) = (2\cos x - \sin x), -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, and $f(\frac{\pi}{3}) = \sqrt{3} + \frac{1}{2}$
8. Suppose the graph of f includes the point (-2, 4) and that the slope of the tangent line to f at x is -2x+4. Find f(5).

In problems #9–10, find the function f that satisfies the given conditions.

9. $f''(x) = \sin x - e^{-2x}$ with $f'(0) = \frac{5}{2}$ and $f(0) = 0$
10. $f''(x) = \frac{1}{\sqrt{x}}$ with $f'(4) = 7$ and $f(4) = 25$

Review Answers

1. $f(x) = 2^{2x} - \frac{4}{3}x^{\frac{3}{2}} + C$
2. $f(x) = -\cos x + \frac{1}{e^x} + C$
3. $f(x) = \frac{8}{15}x^{\frac{5}{2}} + \frac{4}{35}x^{\frac{7}{2}} + C$
4. $f(x) = x^6 - \frac{4}{3}x^3 + \frac{7}{3}x + 2$
5. $f(x) = x^3 + \frac{e^{2x}}{2} + \frac{5}{2}$
6. $f(x) = \frac{3}{5}\sqrt[3]{x^5} + \frac{1}{x} + \frac{7}{5}$
7. $f(x) = 2\sin x + \cos x$
8. $f(x) = -x^2 + 4x + 16$, $f(5) = 11$
9. $f(x) = -\sin x - \frac{1}{4}e^{2x} + 4x + \frac{1}{4}$
10. $f(x) = \frac{4}{3}x^{\frac{3}{2}} + 3x + \frac{7}{3}$

Initial Condition & Integration of Trig Functions Practice

1. Find the particular solution $y = f(x)$ that satisfies the differential equation and initial condition.
   a. $f'(x) = 3\sqrt{x} + 3,\ f(1) = 4$
   b. $f'(x) = 6x(x-1),\ f(10) = -10$
   c. $f'(x) = \dfrac{2-x}{x^3},\ x > 0,\ f(2) = \dfrac{3}{4}$
   d. $f'(x) = \sec^2 x,\ f\left(\dfrac{\pi}{3}\right) = 2\sqrt{3}$

2. Find the equation of the function f whose graph passes through the point.
   $f'(x) = 6\sqrt{x} - 10,\ (4, 2)$

3. Find the function f that satisfies the given conditions.
   a. $f''(x) = 2,\ f'(2) = 5,\ f(2) = 10$
   b. $f''(x) = x^{-2/3},\ f'(8) = 6,\ f(0) = 0$

4. Integrate.
   a. $\int (2\sin x + 3\cos x)\,dx$
   b. $\int (1 - \csc t \cot t)\,dt$
   c. $\int (\csc^2 \theta - \cos\theta)\,d\theta$
   d. $\int (t^2 - \sin t)\,dt$

Answers:

1. a. $f(x) = 2x^{3/2} + 3x - 1$ b. $f(x) = 2x^3 - 3x^2 - 1710$
   c. $f(x) = \dfrac{-1}{x^2} + \dfrac{1}{x} + \dfrac{1}{2}$ d. $f(x) = \tan x + \sqrt{3}$
2. $f(x) = 4x^{3/2} - 10x + 10$
3. a. $f(x) = x^2 + x + 4$ b. $f(x) = \dfrac{9}{4}x^{4/3}$
4. a. $-2\cos x + 3\sin x + C$ b. $t + \csc t + C$
   c. $-\cot\theta - \sin\theta + C$ d. $\dfrac{t^3}{3} + \cos t + C$

7.3 The Area Problem

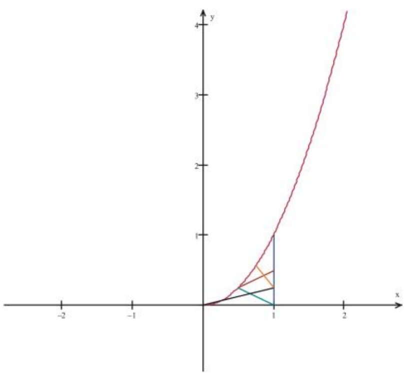
Learning Objectives

- Use sigma notation to evaluate sums of rectangular areas
- Find limits of upper and lower sums
- Use the limit definition of area to solve problems

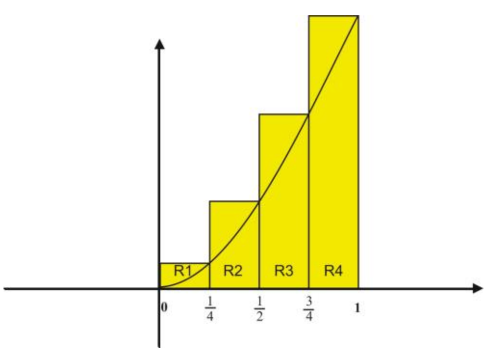
Introduction

In The Lesson The Calculus we introduced the area problem that we consider in integral calculus. The basic problem was this:

$f(x) = x^2$. Suppose we are interested in finding the area between the $x-$axis and the curve of $f(x) = x^2$ from $x = 0$ to $x = 1$.



We approximated the area by constructing four rectangles, with the height of each rectangle equal to the maximum value of the function in the sub-interval.



We then summed the areas of the rectangles as follows:

and $R_1 + R_2 + R_3 + R_4 = \frac{30}{64} = \frac{15}{32} \approx 0.46.$

We call this the ***upper sum*** since it is based on taking the maximum value of the function within each sub-interval. We noted that as we used more rectangles, our area approximation became more accurate.

We would like to formalize this approach for both upper and lower sums. First we note that the ***lower sums*** of the area of the rectangles results in $R_1 + R_2 + R_3 + R_4 = 13/64 \approx 0.20$ Our intuition tells us that the true area lies somewhere between these two sums, or $0.20 < \text{ Area } < 0.46$ and that we will get closer to it by using more and more rectangles in our approximation scheme.

In order to formalize the use of sums to compute areas, we will need some additional notation and terminology.

Sigma Notation

In The Lesson The Calculus we used a notation to indicate the upper sum when we increased our rectangles to $N = 16$ and found that our approximation $A = \sum_1^{16} R_i = \frac{195}{512} \approx 0.38$. The notation we used to enabled us to indicate the sum without the need to write out all of the individual terms. We will make use of this notation as we develop more formal definitions of the area under the curve.

Let's be more precise with the notation. For example, the quantity $A = \sum R_i$ was found by summing the areas of $N = 16$ rectangles. We want to indicate this process, and we can do so by providing indices to the symbols used as follows:

$$A = \sum_{i=1}^{16} R_i = R_1 + R_2 + R_3 + \ldots + R_{15} + R_{16}.$$

The sigma symbol with these indices tells us how the rectangles are labeled and how many terms are in the sum.

Useful Summation Formulas

We can use the notation to indicate useful formulas that we will have occasion to use. For example, you may recall that the sum of the first n integers is $n(n+1)/2$. We can indicate this formula using sigma notation. The formula is given here along with two other formulas that will become useful to us.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

We can show from associative, commutative, and distributive laws for real numbers that

$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n}(a_i) + \sum_{i=1}^{n}(b_i)$ and

$\sum_{i=1}^{n}(ka_i) = k\sum_{i=1}^{n}(a_i).$

Example 1:

Compute the following quantity using the summation formulas:

$$\sum_{i=1}^{10} 2i(i - 6i).$$

Solution:

$$\sum_{i=1}^{10} 2i(i-6i) = \sum_{i=1}^{10}(2i^2 - 12i) = 2\sum_{i=1}^{10} i^2 - 12\sum_{i=1}^{10} i$$
$$= 2\left(\frac{(10)(10+1)(2\cdot 10+1)}{6}\right) - 12\left(\frac{(10)(11)}{2}\right)$$
$$= 770 - 660 = 110.$$

Another Look at Upper and Lower Sums

We are now ready to formalize our initial ideas about upper and lower sums.

Let f be a bounded function in a closed interval $[a, b]$ and $P = [x_0, \ldots, x_n]$ the partition of $[a, b]$ into n subintervals.

We can then define the lower and upper sums, respectively, over partition P, by

$$S(P) = \sum_{1}^{n} m_i(x_i - x_{i-1}) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \ldots + m_n(x_n - x_{n-1}),$$
$$T(P) = \sum_{1}^{n} M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \ldots + M_n(x_n - x_{n-1}).$$

where m_i is the minimum value of f in the interval of length $x_i - x_{i-1}$ and M_i is the maximum value of f in the interval of length $x_i - x_{i-1}$.

The following example shows how we can use these to find the area.

Example 2:

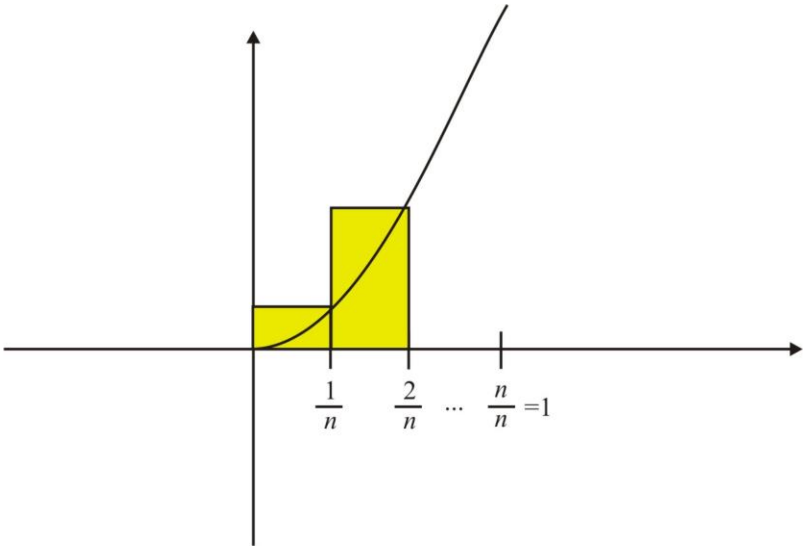
Show that the upper and lower sums for the function $f(x) = x^2$, from $x = 0$ to $x = 1$, approach the value $A = 1/3.$

Solution:

Let P be a partition of n equal sub intervals over $[0, 1]$. We will show the result for the upper sums. By our definition we have

$$T(P) = \sum_{1}^{n} M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \ldots + M_n(x_n - x_{n-1}).$$

We note that each rectangle will have width $\frac{1}{n}$, and lengths $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \ldots, \left(\frac{n}{n}\right)^2$ as indicated:



$$T(P) = \sum_{1}^{n} M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \ldots + M_n(x_n - x_{n-1})$$
$$= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$
$$= \frac{1}{n}\left(\frac{1}{n}\right)^2 (1^2 + 2^2 + 3^2 + \ldots + n^2)$$
$$= \left(\frac{1}{n^3}\right)(1^2 + 2^2 + 3^2 + \ldots + n^2) = \left(\frac{1}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) = \left(\frac{(n+1)(2n+1)}{6n^2}\right).$$

We can re-write this result as:

$$\frac{(n+1)(2n+1)}{6n^2} = \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

We observe that as

$$x \to +\infty, \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \to \frac{1}{3}.$$

We now are able to define the area under a curve as a limit.

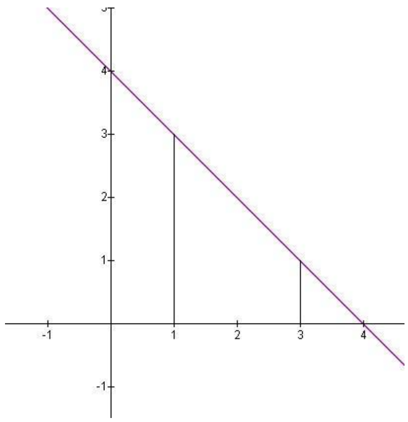
Definition

Let f be a continuous function on a closed interval $[a, b]$. Let P be a partition of n equal sub intervals over $[a, b]$. Then the area under the curve of f is the limit of the upper and lower sums, that is

$$A = \lim_{n \to +\infty} S(P) = \lim_{n \to +\infty} T(P).$$

Example 3:

Use the limit definition of area to find the area under the function $f(x) = 4 - x$ from 1 to $x = 3.$



Solution:

If we partition the interval $[1,3]$ into n equal sub-intervals, then each sub-interval will have length $\frac{3-1}{n} = \frac{2}{n}$ and height $3 - i\triangle x$ as i varies from 1 to n. So we have $\triangle x = \frac{2}{n}$ and

$$S(P) = \sum_{1}^{n}(3 - i\triangle x)\triangle x = \sum_{1}^{n}(3\triangle x) - \sum_{1}^{n} i(\triangle x)^2$$
$$= (3\triangle x)n - \frac{n(n+1)}{n}(\triangle x)^2.$$

Since $\triangle x = \frac{2}{n}$, we then have by substitution

as $n \to \infty$. Hence the area is $A = 4.$

This example may also be solved with simple geometry. It is left to the reader to confirm that the two methods yield the same area.

Lesson Summary

1. We used sigma notation to evaluate sums of rectangular areas.
2. We found limits of upper and lower sums.
3. We used the limit definition of area to solve problems.

Review Questions

In problems #1–2 , find the summations.

1. $\sum_{i=1}^{10} i(2i-3)$
2. $\sum_{i=1}^{n} (3-i)(2+i)$

In problems #3–5, find $S(P)$ and $T(P)$ under the partition P.

3. $f(x) = 1 - x^2, P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$
4. $f(x) = 2x^2, P = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$
5. $f(x) = \frac{1}{x}; P = \{-4, -3, -2, -1\}$

In problems #6–8, find the area under the curve using the limit definition of area.

6. $f(x) = 3x + 5$ from $x = 2$ to $x = 6$.
7. $f(x) = x^2$ from $x = 1$ to $x = 3$.
8. $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 4$.

In problems #9–10, state whether the function is integrable in the given interval. Give a reason for your answer.

9. $f(x) = |x - 2|$ on the interval $[1, 4]$
10. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$, on the interval $[0, 1]$

Review Answers

1. $\sum_{i=1}^{10} i(2i - 3) = 605$
2. $\sum_{i=1}^{n} (3 - i)(2 + i) = \frac{1}{3}(19 - n^2)$
3. $S(P) = -1.75$; $T(P) = 0.25$

   (note that we have included areas under the x-axis as negative values.)

4. $S(P) = 0.5$; $T(P) = 2.5$
5. $S(P) = -1.83$; $T(P) = -1.08$
6. $\text{Area} = 68$
7. $\text{Area} = \frac{26}{3}$
8. $\text{Area} = \frac{15}{16}$
9. Yes, since $f(x) = |x - 2|$ is continuous on $[1, 4]$
10. No, since $S(P) = -1$; $T(P) = 1$

7.4 Definite Integrals

Learning Objectives

- Use Riemann Sums to approximate areas under curves
- Evaluate definite integrals as limits of Riemann Sums

Introduction

In the Lesson The Area Problem we defined the area under a curve in terms of a limit of sums.

$$A = \lim_{n \to +\infty} S(P) = \lim_{n \to +\infty} T(P)$$

where

$$S(P) = \sum_{1}^{n} m_i(x_i - x_{i-1}) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \ldots + m_n(x_n - x_{n-1}),$$

$$T(P) = \sum_{1}^{n} M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \ldots + M_n(x_n - x_{n-1}),$$

$S(P)$ and $T(P)$ were examples of ***Riemann Sums***. In general, Riemann Sums are of form $\sum_{i=1}^{n} f(x_i^*)\triangle x$ where each x_i^* is the value we use to find the length of the rectangle in the i^{th} sub-interval. For example, we used the maximum function value in each sub-interval to find the upper sums and the minimum function in each sub-interval to find the lower sums. But since the function is continuous, we could have used any points within the sub-intervals to find the limit. Hence we can define the most general situation as follows:

Definition

If f is continuous on $[a, b]$ we divide the interval $[a, b]$ into n sub-intervals of equal width with $\triangle x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these sub-intervals and let $x_1^*, x_2^*, \ldots, x_n^*$ be *any* sample points in these sub-intervals. Then the ***definite integral*** of f from $x = a$ to $x = b$ is

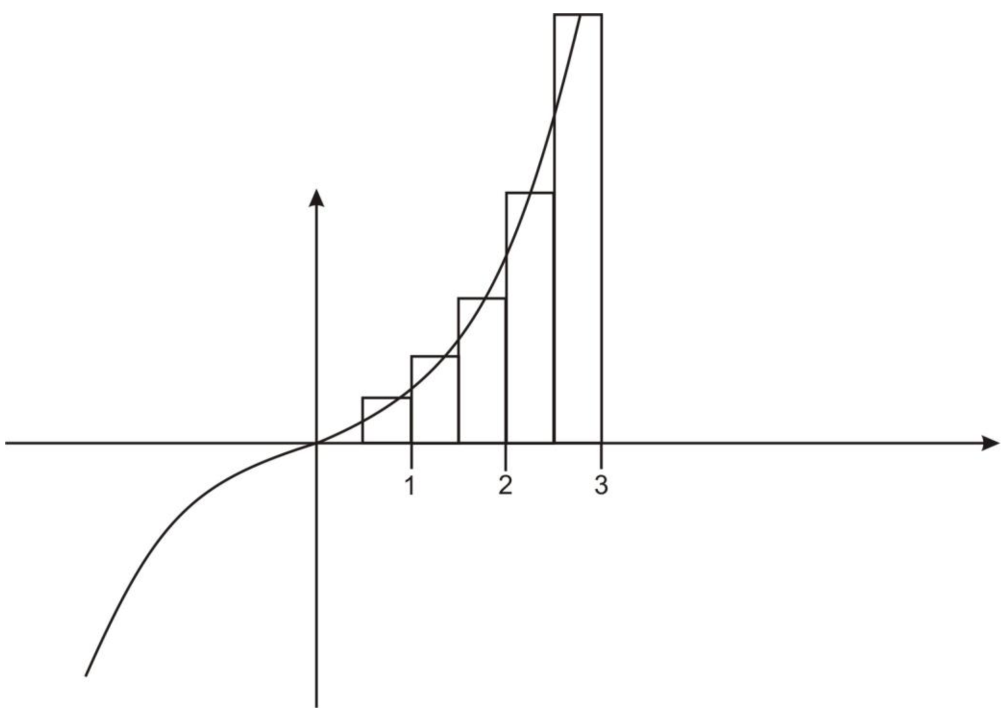
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x.$$

Example 1:

Evaluate the Riemann Sum for $f(x) = x^3$ from $x = 0$ to $x = 3$ using $n = 6$ sub-intervals and taking the sample points to be the midpoints of the sub-intervals.

Solution:

If we partition the interval $[0, 3]$ into $n = 6$ equal sub-intervals, then each sub-interval will have length $\frac{3-0}{6} = \frac{1}{2}$. So we have $\triangle x = \frac{1}{2}$ and



$$\begin{aligned}R_6 &= \sum_1^6 f(x_i^*)\triangle x = f(0.25)\triangle x + f(0.75)\triangle x + f(1.25)\triangle x + f(1.75)\triangle x + f(2.25)\triangle x + f(2.75)\triangle x \\ &= \left(\frac{1}{64}\right)\left(\frac{1}{2}\right) + \left(\frac{27}{64}\right)\left(\frac{1}{2}\right) + \left(\frac{125}{64}\right)\left(\frac{1}{2}\right) + \left(\frac{343}{64}\right)\left(\frac{1}{2}\right) + \left(\frac{729}{64}\right)\left(\frac{1}{2}\right) + \left(\frac{1331}{64}\right)\left(\frac{1}{2}\right) \\ &= \frac{2556}{64} = 39.93.\end{aligned}$$

Now let's compute the definite integral using our definition and also some of our summation formulas.

Example 2:

Use the definition of the definite integral to evaluate $\int_0^3 x^3 dx$.

Solution:

Applying our definition, we need to find

$$\int_0^3 x^3 dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\triangle x.$$

We will use right endpoints to compute the integral. We first need to divide $[0,3]$into $n$sub-intervals of length $\triangle x = \frac{3-0}{n} = \frac{3}{n}$. Since we are using right endpoints, $x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{6}{n}, \ldots x_i = \frac{3i}{n}$.

So $\int_0^3 x^3 dx =$

Recall that $\sum_1^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$. By substitution, we have

$$\int_0^3 x^3 dx = \lim_{n\to\infty} = \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty} \frac{81}{4}\left[1 + \frac{1}{n}\right]^2 \to \frac{81}{4} \text{ as } n \to \infty.$$

Hence

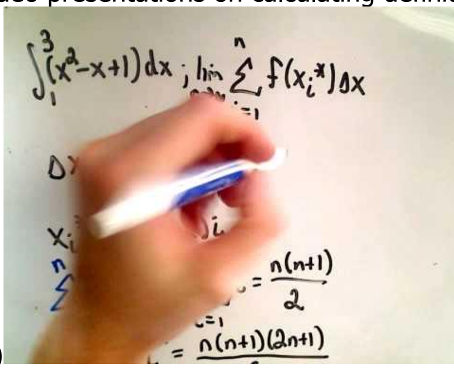
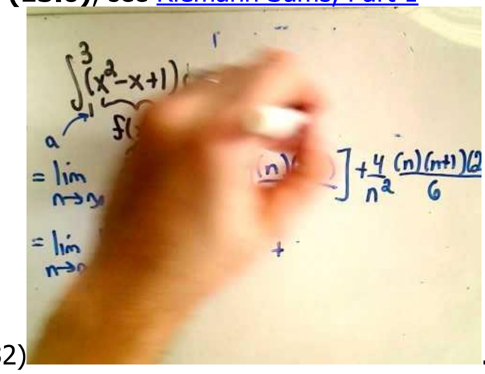
$$\int_0^3 x^3 dx = \frac{81}{4}.$$

Before we look to try some problems, let's make a couple of observations. First, we will soon not need to rely on the summation formula and Riemann Sums for actual computation of definite integrals. We will develop several computational strategies in order to solve a variety of problems that come up. Second, the idea of definite integrals as approximating the area under a curve can be a bit confusing since we may sometimes get results that do not make sense when interpreted as areas. For example, if we were to compute the definite integral $\int_{-3}^3 x^3 dx$, then due to the symmetry of $f(x) = x^3$about the origin, we would find that $\int_{-3}^3 x^3 dx = 0$. This is because for every sample point x_j^*, we also have $-x_j^*$is also a sample point with $f(-x_j^*) = -f(x_j^*)$. Hence, it is more accurate to say that $\int_{-3}^3 x^3 dx$gives us the ***net area*** between $x = -3$and $x = 3$.If we wanted the ***total area*** bounded by the graph and the $x-$axis, then we would compute $2\int_0^3 x^3 dx = \frac{81}{2}$.

Lesson Summary

1. We used Riemann Sums to approximate areas under curves.
2. We evaluated definite integrals as limits of Riemann Sums.

Multimedia Link

For video presentations on calculating definite integrals using Riemann Sums **(13.0)**, see Riemann Sums, Part 1 (6:15) and Riemann Sums, Part 2 (8:32).

The following applet lets you explore Riemann Sums of any function. You can change the bounds and the number of partitions. Follow the examples given on the page, and then use the applet to explore on your own. Riemann Sums Applet. Note: On this page the author uses Left- and Right- hand sums. These are similar to the sums $S(P)$ and $T(P)$ that you have learned, particularly in the case of an increasing (or decreasing) function. Left-hand and Right-hand sums are frequently used in calculations of numerical integrals because it is easy to find the left and right endpoints of each interval, and much more difficult to find the max/min of the function on each interval. The difference is not always important from a numerical approximation standpoint; as you increase the number of partitions, you should see the Left-hand and Right-hand sums converging to the same value. Try this in the applet to see for yourself.

Review Questions

In problems #1–7 , use Riemann Sums to approximate the areas under the curves.

1. Consider $f(x) = 2 - x$ from $x = 0$ to $x = 2.$ Use Riemann Sums with four subintervals of equal lengths. Choose the midpoints of each subinterval as the sample points.
2. Repeat problem #1 using geometry to calculate the exact area of the region under the graph of $f(x) = 2 - x$ from $x = 0$ to $x = 2.$ (Hint: Sketch a graph of the region and see if you can compute its area using area measurement formulas from geometry.)
3. Repeat problem #1 using the definition of the definite integral to calculate the exact area of the region under the graph of $f(x) = 2 - x$ from $x = 0$ to $x = 2.$
4. $f(x) = x^2 - x$ from $x = 1$ to $x = 4.$ Use Riemann Sums with five subintervals of equal lengths. Choose the left endpoint of each subinterval as the sample points, or use trapezoids.
5. Repeat problem #4 using the definition of the definite integral to calculate the exact area of the region under the graph of $f(x) = x^2 - x$ from $x = 1$ to $x = 4.$
6. Consider $f(x) = 3x^2.$ Compute the Riemann Sum of f on [0, 1] under each of the following situations. In each case, use the right endpoint as the sample points, or use trapezoids.
   a. Two sub-intervals of equal length.
   b. Five sub-intervals of equal length.
   c. Ten sub-intervals of equal length.
   d. Based on your answers above, try to guess the exact area under the graph of f on [0, 1].
7. Consider $f(x) = e^x.$ Compute the Riemann Sum of f on [0, 1] under each of the following situations. In each case, use the right endpoint as the sample points.
   a. Two sub-intervals of equal length.
   b. Five sub-intervals of equal length.

c. Ten sub-intervals of equal length.
d. Based on your answers above, try to guess the exact area under the graph of f on [0, 1].

8. Find the net area under the graph of $f(x) = x^3 - x$; $x = -1$ to $x = 1$. (Hint: Sketch the graph and check for symmetry.)
9. Find the total area bounded by the graph of $f(x) = x^3 - x$ and the x-axis, from $x = -1$ to $x = 1$.
10. Use your knowledge of geometry to evaluate the following definite integral: $\int_0^3 \sqrt{9 - x^2}dx$ (Hint: set $y = \sqrt{9 - x^2}$ and square both sides to see if you can recognize the region from geometry.)

Review Answers

1. $\text{Area} = 2$
2. $\text{Area} = 2$
3. $\text{Area} = 2$
4. $\text{Area} = 10.08$ using the Left sum, Area = 13.68 using the Trapezoid Sum
5. $\text{Area} = 15.5$
6.
   a. $\text{Area} = 1.875$ using the Right sum, Area = 1.125 using the Trapezoid Sum
   b. $\text{Area} = 1.32$ using the Right sum, Area = 1.02 using the Trapezoid Sum
   c. $\text{Area} = 1.15$ using the Right sum, Area = 1.005 using the Trapezoid Sum
   d. $\text{Area} = 1$ is a good guess
7.
   a. $\text{Area} = 2.18$
   b. $\text{Area} = 1.89$
   c. $\text{Area} = 1.80$
   d. $\text{Area} = e^1 - 1 \approx 1.71$
8. The graph is symmetric about the origin; hence net Area = 0
9. Area = 1/2
10. The graph is that of a quarter circle of radius 3; hence Area = $\frac{9\pi}{4}$

7.5 Evaluating Definite Integrals

Learning Objectives

- Use antiderivatives to evaluate definite integrals
- Use the Mean Value Theorem for integrals to solve problems
- Use general rules of integrals to solve problems

Introduction

In the Lesson on Definite Integrals, we evaluated definite integrals using the limit definition. This process was long and tedious. In this lesson we will learn some practical ways to evaluate definite integrals. We begin with a theorem that provides an easier method for evaluating definite integrals. Newton discovered this method that uses antiderivatives to calculate definite integrals.

Theorem:

If f is continuous on the closed interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is any antiderivative of f.

We sometimes use the following shorthand notation to indicate $\int_a^b f(x)dx = F(b) - F(a)$:

$$\int_a^b f(x)dx = F(x)\Big]_a^b.$$

The proof of this theorem is included at the end of this lesson. Theorem 4.1 is usually stated as a part of the Fundamental Theorem of Calculus, a theorem that we will present in the Lesson on the Fundamental Theorem of Calculus. For now, the result provides a useful and efficient way to compute definite integrals. We need only find an antiderivative of the given function in order to compute its integral over the closed interval. It also gives us a result with which we can now state and prove a version of the Mean Value Theorem for integrals. But first let's look at a couple of examples.

Example 1:

Compute the following definite integral:

$$\int_0^3 x^3 dx.$$

Solution:

Using the limit definition we found that $\int_0^3 x^3 dx = \frac{81}{4}$. We now can verify this using the theorem as follows:

We first note that $x^4/4$ is an antiderivative of $f(x) = x^3$. Hence we have

$$\int_0^3 x^3 dx = \frac{x^4}{4}\Big]_0^3 = \frac{81}{4} - \frac{0}{4} = \frac{81}{4}.$$

We conclude the lesson by stating the rules for definite integrals, most of which parallel the rules we stated for the general indefinite integrals.

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \text{ where } a < c < b.$$

Given these rules together with Theorem 4.1, we will be able to solve a great variety of definite integrals.

Example 2:

Compute $\int_{-2}^{2} (x - \sqrt{x})dx.$

Solution:

$$\int_1^4 (x-\sqrt{x})dx = \int_1^4 x dx - \int_1^4 \sqrt{x} dx = \frac{x^2}{2}\Big]_1^4 - \frac{2}{3}x^{\frac{3}{2}}\Big]_1^4 = \left(8-\frac{1}{2}\right) - \frac{2}{3}(8-1) = \frac{15}{2} - \frac{14}{3} = \frac{17}{6}.$$

Example 3:

Compute $\int_0^{\frac{\pi}{2}} (x+\cos x)dx.$

Solution:

$$\int_0^{\frac{\pi}{2}} (x+\cos x)dx = \int_0^{\frac{\pi}{2}} (x)dx = \int_0^{\frac{\pi}{2}} (\cos x)dx = \frac{x^2}{2}\Big]_0^{\frac{\pi}{2}} + \frac{\sin x}{1}\Big]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4} + 1 = \frac{\pi^2+4}{4}.$$

Lesson Summary

1. We used antiderivatives to evaluate definite integrals.
2. We used the Mean Value Theorem for integrals to solve problems.
3. We used general rules of integrals to solve problems.

Proof of Theorem 4.1

We first need to divide $[a,b]$into $n$sub-intervals of length $\triangle x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \ldots, x_n = b$be the endpoints of these sub-intervals.

Let Fbe any antiderivative of f.

Consider $F(b) - F(a) = F(x_n) - F(x_0).$

We will now employ a method that will express the right side of this equation as a Riemann Sum. In particular,

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) - F(x_{n-1}) + F(x_{n+1}) - F(x_{n-2}) + F(x_{n-2}) - \ldots - F(x_1) + F(x_1) \\ &= \sum_1^n [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Note that $F$is continuous. Hence, by the Mean Value Theorem, there exist $c_i \in [x_i - 1, x_i]$

such that $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\triangle x.$

Hence

$$F(b) - F(a) = \sum_1^n F'(c_i)(x_i - x_{i-1}) = \sum_1^n f(c_i)\triangle x.$$

Taking the limit of each side as $n \to \infty$we have

$$\lim_{n\to\infty}[F(b)-F(a)] = \lim_{n\to\infty}\sum_1^n f(c_i)\triangle x.$$

We note that the left side is a constant and the right side is our definition for $\int_a^b f(x)dx$.

Hence

$$F(b)-F(a) = \lim_{n\to\infty}\sum_1^n f(c_i)\triangle x = \int_a^b f(x)dx.$$

Proof of Theorem 4.2

Let $F(x) = \int_a^x f(x)dx.$

By the Mean Value Theorem for derivatives, there exists $c \in [a,b]$such that

$$F'(c) = \frac{F(b)-F(a)}{b-a}.$$

From Theorem 4.1 we have that $F$is an antiderivative of f.Hence, $F'(x) = f(x)$and in particular, $F'(c) = f(c).$ Hence, by substitution we have $f(c) = \frac{F(b)-F(a)}{b-a}.$

Note that $F(a) = \int_a^a f(x)dx = 0.$ Hence we have $f(c) = \frac{F(b)-0}{b-a} = \frac{F(b)}{b-a},$

and by our definition of $F(x)$ we have $f(c) = \frac{1}{b-a}F(b) = \frac{1}{b-a}\int_a^b f(x)dx.$

This theorem allows us to find for positive functions a rectangle that has base $[a,b]$and height $f(c)$such that the area of the rectangle is the same as the area given by $\int_a^b f(x)dx.$In other words, $f(c)$is the average function value over $[a,b].$

Review Questions

In problems #1–8, use antiderivatives to compute the definite integral.

1. $\int_4^9 (\frac{3}{\sqrt{x}})dx$
2. $\int_0^1 (t-t^2)dt$
3. $\int_2^5 (\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{2}})dx$
4. $\int_0^1 4(x^2-1)(x^2+1)dx$
5. $\int_2^8 (\frac{4}{x}+x^2+x)dx$
6. $\int_2^4 (e^{3x})dx$
7. $\int_1^4 \frac{2}{x+3}dx$
8. Find the average value of $f(x) = \sqrt{x}$ over [1, 9].
9. If f is continuous and $\int_1^4 f(x)dx = 9,$ show that f takes on the value 3 at least once on the interval [1, 4].

10. Your friend states that there is no area under the curve of $f(x) = \sin x$ on $[0, 2\pi]$ since he computed $\int_0^{2\pi} \sin x dx = 0.$ Is he correct? Explain your answer.

Review Answers

1. $\int_4^9 (\frac{3}{\sqrt{x}}) dx = 6$
2. $\frac{1}{6}$
3. $\int_2^5 (\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{2}}) dx = 2\sqrt{5} - 2\sqrt{2} + \frac{3\sqrt{2}}{2} = 2\sqrt{5} - \frac{\sqrt{2}}{2}$
4. $\int_0^1 4(x^2 - 1)(x^2 + 1) dx = -\frac{16}{5}$
5. $\int_2^8 (\frac{4}{x} + x^2 + x) dx = \frac{9417}{48} \approx 196.19$
6. $\int_2^4 (e^{3x}) dx = \frac{e^{12} - e^6}{3}$
7. $\int_1^4 \frac{2}{x+3} dx = 2\ln 7 - 2\ln 4$
8. $\frac{13}{6}$
9. Apply the Mean Value Theorem for integrals.
10. He is partially correct. The definite integral $\int_0^{2\pi} \sin x \, dx$ computes the **net** area under the curve. However, the area between the curve and the x-axis is given by: $A = 2\int_0^{\pi} \sin x \, dx = 2\left[-\cos x\right]_0^{\pi} = 4$

Practice with the Mean Value Theorem, Average Value & Properties of Integrals

Find the average value of the function over the interval. Then find all x-values in the interval for which the function is equal to its average value.

1. $f(x)=6-x^2, [-2,2]$

2. $f(x)=x\sqrt{4-x^2}, [0,2]$

3. A deposit of $2250 is made in a savings account with an annual interest rate of 12%, compounded continuously. Find the average balance in the account during the first 5 years.

4. In the United States, the annual death rate R (in deaths per 1000 people x years old) can be modeled by $R=.036x^2-2.8x+58.14,\ 40\le x\le 60$. Find the average death rate for people between a) 40 and 50 years of age, and b) 50 and 60 years of age.

5. Use the value $\int_0^2 x^2dx=\frac{8}{3}$ to evaluate the definite integral.

a. $\int_{-2}^0 x^2dx$ b. $\int_{-2}^2 x^2dx$ c. $\int_0^2 -x^2dx$

6. Use the value $\int_0^2 x^3dx=4$ to evaluate the definite integral.

a. $\int_{-2}^0 x^3dx$ b. $\int_{-2}^2 x^3dx$ c. $\int_0^2 3x^3dx$

7. Use the values $\int_0^5 f(x)dx=8$ and $\int_0^5 g(x)dx=3$ to evaluate the definite integral.

a. $\int_0^5[f(x)+g(x)]dx$ b. $\int_0^5[f(x)-g(x)]dx$ c. $\int_0^5[f(x)-f(x)]dx$

d. $\int_0^5 -4f(x)dx$ e. $\int_0^5[f(x)-3g(x)]dx$ f. $\int_5^5 f(x)dx$ g. $\int_5^0 f(x)dx$

Answers:

1. average value = $\frac{14}{3}$

$x=\frac{\pm2\sqrt{3}}{3}\approx\pm1.155$

2. average value = $\frac{4}{3}$

$x=\sqrt{2+\frac{2\sqrt{5}}{3}}\approx1.868$

$x=\sqrt{2-\frac{2\sqrt{5}}{3}}\approx.714$

3. $3082.95
4. a) 5.34 deaths per 1000 people b) 13.34 deaths per 1000 people
5. a. $\frac{8}{3}$ b. $\frac{16}{3}$ c. $\frac{-8}{3}$
6. a. -4 b. 0 c. 12
7. a. 11 b. 5 c. 0
d. -32 e. -1 f. 0 g. -8

Integral Properties & Average Value HW

Use the given info and the properties of integrals to evaluate the requested definite integrals.

1.) Given: $\int_0^2 f(x)\,dx = \frac{8}{3}$, and $f(x)$ is an even function

a.) $\int_{-2}^0 f(x)\,dx =$ b.) $\int_{-2}^2 f(x)\,dx =$ c.) $\int_0^2 5f(x)\,dx =$

2.) Given: $\int_0^2 g(x)\,dx = 4$, and $f(x)$ is an odd function

a.) $\int_{-2}^0 g(x)\,dx =$ b.) $\int_{-2}^2 g(x)\,dx =$ c.) $\int_0^2 5g(x)\,dx =$

3.) Given: $\int_0^5 f(x)\,dx = 8$ and $\int_0^5 g(x)\,dx = 3$

a.) $\int_0^5 [f(x)+g(x)]\,dx =$ b.) $\int_0^5 [f(x)-g(x)]\,dx =$ c.) $\int_0^5 [f(x)-f(x)]\,dx =$

d.) $\int_0^5 [f(x)-3g(x)]\,dx =$ e.) $\int_5^5 f(x)\,dx =$ f.) $\int_5^0 f(x)\,dx =$

g.) $\int_0^5 [f(x)g(x)]\,dx =$ h.) $\int_0^5 \left[\frac{f(x)}{g(x)}\right]dx =$

Find the average value of the function over the given interval, and find the *x*-value in the interval where the function takes on the average value.

4.) $f(x) = 6 - x^2$, [–2, 2]

5.) $f(x) = x\sqrt{4-x^2}$, [0, 2]

Answers:

1.) 8/3, 16/3, 40/3
2.) –4, 0, 20
3.) 11, 5, 0, –1, 0, –8, g & h cannot be done with the given info – multiplying / dividing areas does not yield areas!

4.) AV = 14/3, $x = \pm 1.155 \in [-2, 2]$
5.) AV = 4/3, $x = .714, 1.868 \in [0, 2]$

7.6 The Fundamental Theorem of Calculus

Learning Objectives

- Use the Fundamental Theorem of Calculus to evaluate definite integrals

Introduction

In the Lesson on Evaluating Definite Integrals, we evaluated definite integrals using antiderivatives. This process was much more efficient than using the limit definition. In this lesson we will state the Fundamental Theorem of Calculus and continue to work on methods for computing definite integrals.

Fundamental Theorem of Calculus:

Let f be continuous on the closed interval $[a, b]$.

1. If function F is defined by $F(x) = \int_a^x f(t)dx$ on $[a, b]$, then $F'(x) = f(x)$ on $[a, b]$.
2. If g is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(t)dt = g(b) - g(a).$$

We first note that we have already proven part 2 as Theorem 4.1. The proof of part 1 appears at the end of this lesson.

Think about this Theorem. Two of the major unsolved problems in science and mathematics turned out to be solved by calculus which was invented in the seventeenth century. These are the ancient problems:

1. Find the areas defined by curves, such as circles or parabolas.
2. Determine an instantaneous rate of change or the slope of a curve at a point.

With the discovery of calculus, science and mathematics took huge leaps, and we can trace the advances of the space age directly to this Theorem.

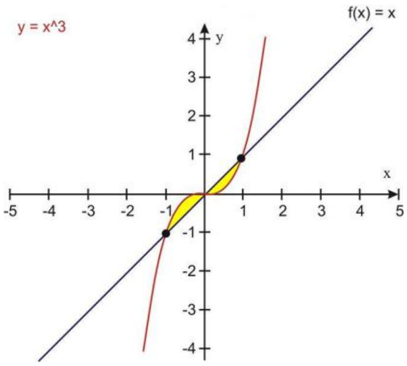
Let's continue to develop our strategies for computing definite integrals. We will illustrate how to solve the problem of finding the area bounded by two or more curves.

Example 1:

Find the area between the curves of $f(x) = x$ and $g(x) = x^3$.

Solution:

We first observe that there are no limits of integration explicitly stated here. Hence we need to find the limits by analyzing the graph of the functions.



We observe that the regions of interest are in the first and third quadrants from $x = -1$ to $x = 1$. We also observe the symmetry of the graphs about the origin. From this we see that the total area enclosed is

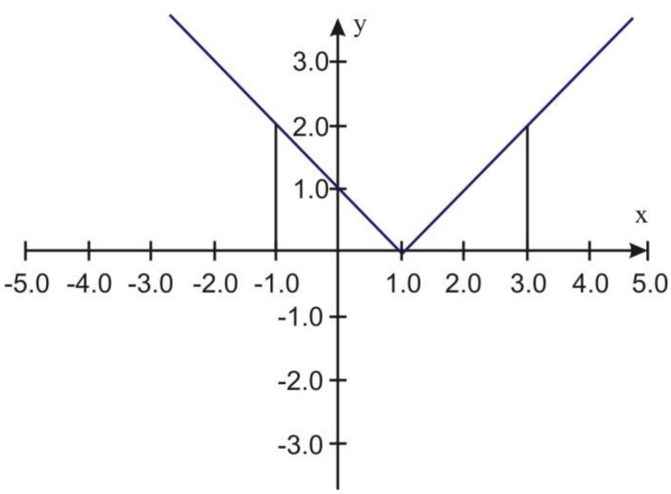
$$2\int_0^1 (x - x^3)dx = 2\left[\int_0^1 x\,dx - \int_0^1 x^3 dx\right] = 2\left[\frac{x^2}{2}\Big|_0^1 - \frac{x^4}{4}\Big|_0^1 =\right] = 2\left[\frac{1}{2} - \frac{1}{4}\right] = 2\left[\frac{1}{4}\right] = \frac{1}{2}.$$

Example 2:

Find the area between the curves of $f(x) = |x - 1|$ and the $x-$axis from $x = -1$ to $x = 3$.

Solution:

We observe from the graph that we will have to divide the interval $[-1, 3]$ into subintervals $[-1, 1]$ and $[1, 3]$.



Hence the area is given by

$$\int_{-1}^{1}(-x+1)dx + \int_1^3 (x-1)dx = \left(-\frac{x^2}{2} + x\right)\Big|_{-1}^{+1} + \left(\frac{x^2}{2} - x\right)\Big|_{+1}^{+3} = 2 + 2 = 4.$$

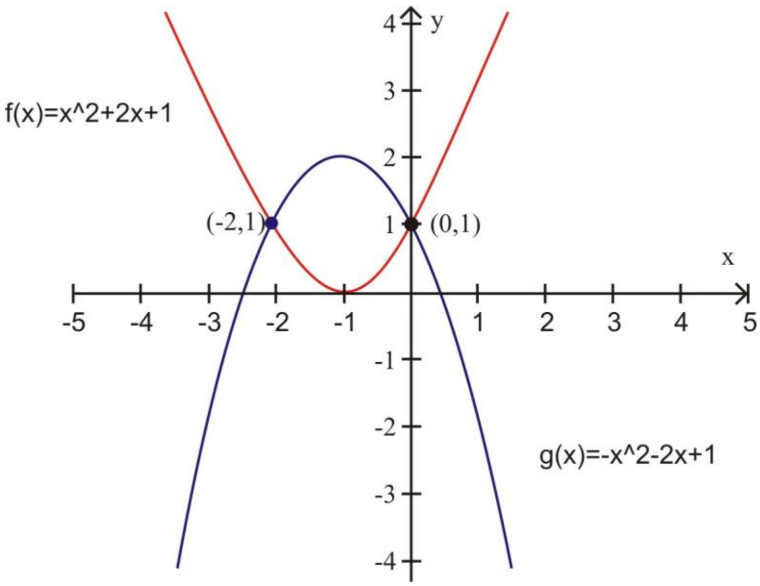
Example 3:

Find the area enclosed by the curves of $f(x) = x^2 + 2x + 1$ and

$g(x) = -x^2 - 2x + 1.$

Solution:

The graph indicates the area we need to focus on.



$$\int_{-2}^{0}(-x^2 - 2x + 1)dx - \int_{-2}^{0}(x^2 + 2x + 1)dx = \left(-\frac{x^3}{3} - x^2 + x\right)\Big|_{-2}^{0} + \left(\frac{x^2}{3} - x^3 + x\right)\Big|_{-2}^{0} = \frac{10}{3} - \frac{2}{3} = \frac{8}{3}$$

Before providing another example, let's look back at the first part of the Fundamental Theorem. If function F is defined by $F(x) = \int_a^x f(t)dt$, on $[a, b]$ then $F'(x) = f(x)$ on $[a, b]$. Observe that if we differentiate the integral with respect to x, we have

$$\frac{d}{dx}\int_a^x f(t)dt = F'(x) = f(x).$$

This fact enables us to compute derivatives of integrals as in the following example.

Example 4:

Use the Fundamental Theorem to find the derivative of the following function:

$$g(x) = \int_0^x (1 + \sqrt[3]{t})dt.$$

Solution:

While we could easily integrate the right side and then differentiate, the Fundamental Theorem enables us to find the answer very routinely.

$$g'(x) = \frac{d}{dx}\int_0^x (1 + \sqrt[3]{t})dt = 1 + \sqrt[3]{x}.$$

This application of the Fundamental Theorem becomes more important as we encounter functions that may be more difficult to integrate such as the following example.

Example 5:

Use the Fundamental Theorem to find the derivative of the following function:

$$g(x) = \int_2^x (t^2 \cos t)dt.$$

Solution:

In this example, the integral is more difficult to evaluate. The Fundamental Theorem enables us to find the answer routinely.

$$g'(x) = \frac{d}{dx} \int_2^x (t^2 \cos t)dt = x^2 \cos x.$$

Lesson Summary

1. We used the Fundamental Theorem of Calculus to evaluate definite integrals.

Fundamental Theorem of Calculus

Let f be continuous on the closed interval $[a, b]$.

1. If function F is defined by $F(x) = \int_a^x f(t)dt$, on $[a, b]$, then $F'(x) = f(x)$, on $[a, b]$.
2. If g is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(t)dt = g(b) - g(a).$$

We first note that we have already proven part 2 as Theorem 4.1.

Proof of Part 1.

1. Consider $F(x) = \int_a^x f(t)dt$, on $[a, b]$.

2. $x, c \in [a, b]$, $c < x$.

Then $\int_a^x f(t)dt = \int_a^c f(t)dt + \int_c^x f(t)dt$ by our rules for definite integrals.

3. Then $\int_a^x f(t)dt - \int_a^c f(t)dt = \int_c^x f(t)dt$. Hence $F(x) - F(c) = \int_c^x f(t)dt.$

4. Since f is continuous on $[a, b]$ and $x, c \in [a, b]$, $c < x$ then we can select $u, v \in [c, v]$ such that $f(u)$ is the minimum value of and $f(v)$ is the maximum value of f in $[c, x]$. Then we can consider $f(u)(x - c)$ as a lower sum and $f(v)(x - c)$ as an upper sum of f from c to x. Hence

5. $f(u)(x - c) \leq \int_c^x f(t)dt \leq f(v)(x - c).$

6. By substitution, we have:

$$f(u)(x-c) \leq F(x) - F(c) \leq f(v)(x-c).$$

7. By division, we have

$$f(u) \leq \frac{F(x)-F(c)}{x-c} \leq f(v).$$

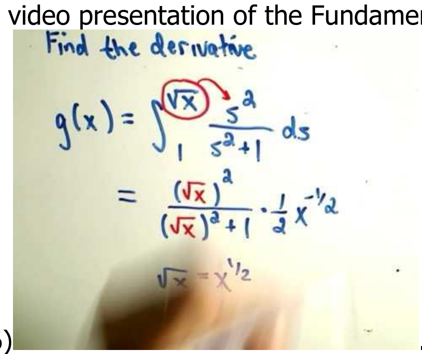
8. When x is close to c, then both $f(u)$ and $f(v)$ are close to $f(c)$ by the continuity of f

9. Hence $\lim_{x\to c^+} \frac{F(x)-F(c)}{x-c} = f(c)$. Similarly, if $x < c$, then $\lim_{x\to c^-} \frac{F(x)-F(c)}{x-c} = f(c)$. Hence, $\lim_{x\to c} \frac{F(x)-F(c)}{x-c} = f(c)$.

10. By the definition of the derivative, we have that

$F'(c) = \lim_{x\to c} \frac{F(x)-F(c)}{x-c} = f(c)$ for every $c \in [a,b]$. Thus, F is an antiderivative of f on $[a,b]$.

Multimedia Link

For a video presentation of the Fundamental Theorem of Calculus **(15.0)**, see Fundamental Theorem of Calculus, Part 1 (9:26)

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Review Questions

In problems #1–4, sketch the graph of the function f(x) in the interval [a, b]. Then use the Fundamental Theorem of Calculus to find the area of the region bounded by the graph and the x-axis. (Hint: Examine the graph of the function and divide the interval accordingly.)

1. $f(x) = 2x+3, [0,4]$
2. $f(x) = e^x, [0,2]$
3. $f(x) = x^2 + x, [1,3]$
4. $f(x) = x^2 - x, [0,2]$

In problems #5–7 use antiderivatives to compute the definite integral. (Hint: Examine the graph of the function and divide the interval accordingly.)

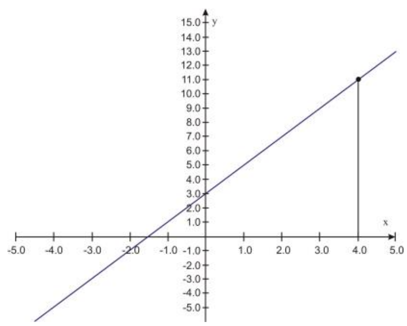
5. $\int_{-1}^{+1} |x| dx$
6. $\int_0^3 |x^3 - 1| dx$
7. $\int_{-2}^{+4} [|x-1| + |x+1|] dx$

In problems #8–10, find the area between the graphs of the functions.

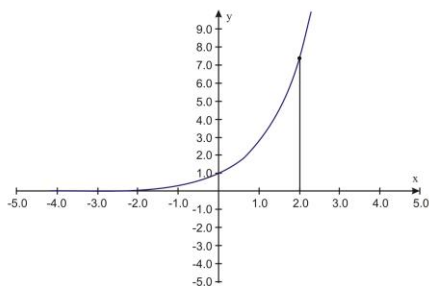
8. $f(x) = \sqrt{x}, g(x) = x$
9. $f(x) = x^2, g(x) = 4$
10. $f(x) = x^2 + 1, g(x) = 3 - x$, on the interval (0, 3) (Hint: you will need to add 2 integrals)

Review Answers

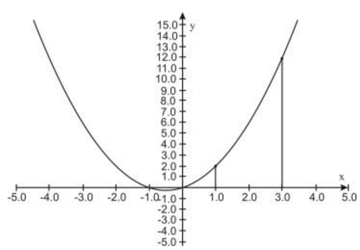
1. $\text{Area} = 28$



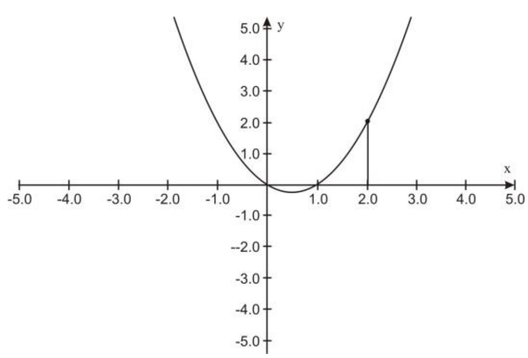
2. $\text{Area} = e^2 - 1$



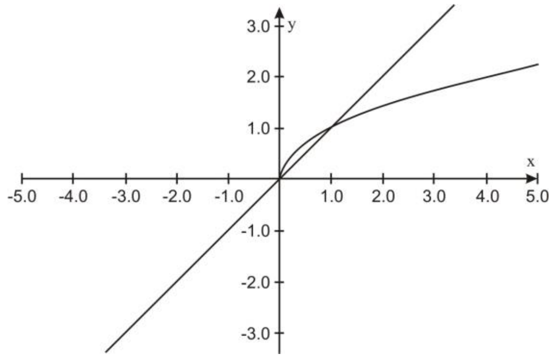
3. $\text{Area} = \frac{38}{3}$



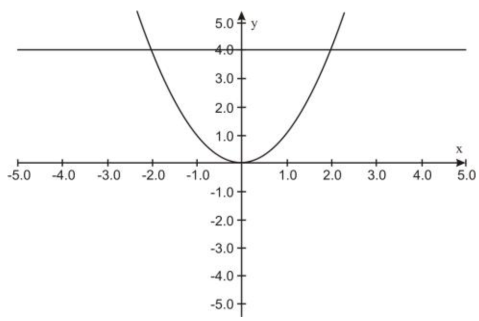
4. $\text{Area} = 1$



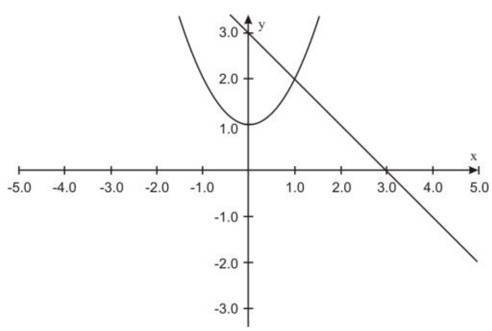
5. $\int_{-1}^{+1} |x| dx = 1$
6. 18.75
7. $\int_{-2}^{+4} \left[|x-1| + |x+1| \right] dx = 22$
8. Area = 1/6



9. Area = 32/3



10. Area = $\frac{59}{6}$



7.7 Integration by Substitution

Learning Objectives

- Integrate composite functions
- Use change of variables to evaluate definite integrals
- Use substitution to compute definite integrals

Introduction

In this lesson we will expand our methods for evaluating definite integrals. We first look at a couple of situations where finding antiderivatives requires special methods. These involve finding antiderivatives of composite functions and finding antiderivatives of products of functions.

Antiderivatives of Composites

Suppose we needed to compute the following integral:

$$\int 3x^2\sqrt{1+x^3}dx.$$

Our rules of integration are of no help here. We note that the integrand is of the form $f(g(x)) * g'(x)$ where $g(x) = 1 + x^3$ and $f(x) = \sqrt{x}.$

Since we are looking for an antiderivative F of f, and we know that $F' = f$, we can re-write our integral as

$$\int \sqrt{1+x^3} \cdot 3x^2 dx = \frac{2}{3}(\sqrt{1+x^3})^{\frac{3}{2}} + C.$$

In practice, we use the following substitution scheme to verify that we can integrate in this way:

4. Integrate with respect to u:
$\int \sqrt{1+x^3} \cdot 3x^2 dx = \int \sqrt{u}du$, where $u = 1 + x^3$ and $du = 3x^2 dx.$
3. Change the original integral in x to an integral in u:
2. Differentiate both sides so $du = 3x^2 dx.$
1. Let $u = 1 + x^3.$

$$\int \sqrt{u}du = \int u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

5. Change the answer back to x:

$$\int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(\sqrt{1+x^3})^{\frac{3}{2}} + C.$$

While this method of substitution is a very powerful method for solving a variety of problems, we will find that we sometimes will need to modify the method slightly to address problems, as in the following example.

Example 1:

Compute the following indefinite integral:

$$\int x^2 e^{x^3} dx.$$

Solution:

We note that the derivative of x^3 is $3x^2$; hence, the current problem is not of the form $\int F'(g(x)) \cdot g'(x)dx$. But we notice that the derivative is off only by a constant of 3 and we know that constants are easy to deal with when differentiating and integrating. Hence

Let $u = x^3$.

Then $du = 3x^2 dx$.

Then $\frac{1}{3} du = x^2 dx$. and we are ready to change the original integral from x to an integral in u and integrate:

$$\int x^2 e^{x^3} dx = \int e^u \left(\frac{1}{3} du\right) = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C.$$

Changing back to x, we have

$$\int x^2 e^{x^3} dx = \frac{1}{3} e^{x^3} + C.$$

We can also use this substitution method to evaluate definite integrals. If we attach limits of integration to our first example, we could have a problem such as

$$\int_1^4 \sqrt{1 + x^3} \cdot 3x^2 dx.$$

The method still works. However, we have a choice to make once we are ready to use the Fundamental Theorem to evaluate the integral.

Recall that we found that $\int \sqrt{1 + x^3} \cdot 3x^2 dx = \int \sqrt{u} du$ for the indefinite integral. At this point, we could evaluate the integral by changing the answer back to x or we could evaluate the integral in u. But we need to be careful. Since the original limits of integration were in x, we need to change the limits of integration for the equivalent integral in u. Hence,

$\int_1^4 \sqrt{1 + x^3} \cdot 3x^2 dx = \int_{u=2}^{65} \sqrt{u} du$, where $u = 1 + x^3$

$$\int_1^4 \sqrt{1 + x^3} \cdot 3x^2 dx = \int_{u=2}^{65} \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} \Big|_{u=2}^{u=65} = \frac{2}{3} (\sqrt{65^3} - \sqrt{8}).$$

Integrating Products of Functions

We are not able to state a rule for integrating products of functions, $\int f(x) g(x) dx$ but we can get a relationship that is almost as effective. Recall how we differentiated a product of functions:

$$\frac{d}{dx} f(x) g(x) = f(x) g'(x) + g(x) f'(x).$$

So by integrating both sides we get

$\int [f(x) g'(x) + g(x) f'(x)] dx = f(x) g(x)$, or

$$\int f(x) g'(x) dx = f(x) g(x) - \int g(x) f'(x).$$

In order to remember the formula, we usually write it as

$$\int u dv = uv - \int v du.$$

We refer to this method as integration by parts. The following example illustrates its use.

Example 2:

Use integration by parts method to compute

$$\int xe^x dx.$$

Solution:

We note that our other substitution method is not applicable here. But our integration by parts method will enable us to reduce the integral down to one that we can easily evaluate.

Let $u = x$and $dv = e^x dx$then $du = dx$and $v = e^x$

By substitution, we have

$$\int xe^x dx = xe^x - \int e^x dx.$$

We can easily evaluate the integral and have

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

And should we wish to evaluate definite integrals, we need only to apply the Fundamental Theorem to the antiderivative.

Lesson Summary

1. We integrated composite functions.
2. We used change of variables to evaluate definite integrals.
3. We used substitution to compute definite integrals.

Review Questions

Compute the integrals in problems #1–8.

1. $\int x \ln x dx$
2. $\int \frac{x}{\sqrt{2x+1}} dx$
3. $\int_0^1 x^3 \sqrt{1 - x^2} dx$
4. $\int x \cos x dx$
5. $\int_0^1 x^2 \sqrt{x^3 + 9} dx$
6. $\int \left(\frac{1}{x^2} \cdot e^{\frac{1}{x}} \right) dx$
7. $\int x^3 e^{x^2} dx$
8. $\int_1^e \frac{1}{x} dx$

Review Answers

1. $\int x \ln x dx = \frac{x^2(2\ln x - 1)}{4} + C$
2. $\int \frac{x}{\sqrt{2x+1}} dx = \frac{(x-1)\sqrt{2x+1}}{3} + c$
3. $\int_0^1 x^3\sqrt{1-x^2}dx = \frac{2}{15}$
4. $\int x \cos x dx = x \sin x + \cos x + c$
5. $\int_0^1 x^2\sqrt{x^3+9}dx = \frac{2}{9}\left[10^{\frac{3}{2}} - 27\right]$
6. $\int \left(\frac{1}{x^2} \cdot e^{\frac{1}{x}}\right) dx = -e^{\frac{1}{x}} + c$
7. $\int x^3 e^{x^2} dx = \frac{1}{2}e^{x^2}(x^2 - 1) + c$
8. $\int_1^e \frac{1}{x} dx = 1$

Integration by Substitution Practice

Find the indefinite integral.

1. $\int (x^2-1)^3(2x)\,dx$

2. $\int \sqrt{3-x^3}\,(3x^2)\,dx$

3. $\int (x-3)^{5/2}\,dx$

4. $\int x(1-2x^2)^3\,dx$

5. $\int \frac{x^2}{(x^3-1)^2}\,dx$

6. $\int \frac{6x}{(1+x^2)^3}\,dx$

7. $\int \frac{4x+6}{(x^2+3x+7)^3}\,dx$

8. $\int m^3\sqrt{m^4+2}\,dm$

9. $\int \frac{x^2}{\sqrt{1-x^3}}\,dx$

10. $\int \frac{t+2t^2}{\sqrt{t}}\,dt$

11. $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$

12. $\int \frac{1}{3x^2}\,dx$

13. $\int (3-2x-4x^2)(1+4x)\,dx$

14. Find the equation of the function f whose graph passes through the point $\left(0,\frac{7}{3}\right)$ and whose derivative is $f'(x)=x\sqrt{1-x^2}$.

Answers:

1. $\frac{(x^2-1)^4}{4}+C$

2. $\frac{-2}{3}(3-x^3)^{3/2}+C$

3. $\frac{2}{7}(x-3)^{7/2}+C$

4. $\frac{-1}{16}(1-2x^2)^4+C$

5. $\frac{-1}{3(x^3-1)}+C$

6. $-\frac{3}{2(1+x^2)^2}+C$

7. $\frac{-1}{(x^2+3x+7)^2}+C$

8. $\frac{1}{6}(m^4+2)^{3/2}+C$

9. $\frac{-2}{3}\sqrt{1-x^3}+C$

10. $\frac{2}{15}t^{3/2}(5+6t)+C$

11. $\frac{-1}{4}\left(1+\frac{1}{t}\right)^4+C$

12. $\frac{-1}{3x}+C$

13. $\frac{-1}{4}(3-2x-4x^2)^2+C$

14. $f(x)=\frac{-1}{3}(1-x^2)^{3/2}+\frac{8}{3}$

General Power Rule HW 1

Integrate!

1.) $\int 2x\left(x^2-1\right)^3\,dx$

2.) $\int 3x^2\sqrt{3-x^3}\,dx$

3.) $\int \left(x-3\right)^{\frac{5}{2}}\,dx$

4.) $\int x\left(1-2x^2\right)^3\,dx$

5.) $\int \dfrac{x^2}{\left(x^3-1\right)^2}\,dx$

6.) $\int \dfrac{4x+6}{\left(x^2+3x+7\right)^3}\,dx$

7.) Find the equation of the function $f(x)$ whose graph passes through the point $\left(0,\dfrac{7}{3}\right)$ and whose derivative is $f'(x)=x\sqrt{1-x^2}$.

Answers:

1.) $\dfrac{\left(x^2-1\right)^4}{4}+C$

2.) $-\dfrac{2}{3}\left(3-x^3\right)^{\frac{3}{2}}+C$

3.) $\dfrac{2}{7}\left(x-3\right)^{\frac{7}{2}}+C$

4.) $-\dfrac{1}{16}\left(1-2x^2\right)^4+C$

5.) $-\dfrac{1}{3\left(x^3-1\right)}+C$

6.) $-\dfrac{1}{\left(x^2+3x+7\right)^2}+C$

7.) $f(x)=-\dfrac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}+\dfrac{8}{3}$

General Power Rule HW 2

Integrate! Some of these require the general power rule (substitution), others do not.

1.) $\int \frac{6x}{\left(1+x^2\right)^3}\,dx$

2.) $\int \frac{t+2t^2}{\sqrt{t}}\,dt$

3.) $\int \left(3-2x-4x^2\right)\left(1+4x\right)dx$

4.) $\int \frac{1}{3x^2}\,dx$

5.) $\int 5x\sqrt[3]{1-x^2}\,dx$

6.) $\int (x+5)^6\,dx$

7.) $\int (3x-8)^{10}\,dx$

Answers:

1.) $-\frac{3}{2\left(1+x^2\right)^2}+C$

2.) $\frac{2}{15}t^{\frac{3}{2}}\left(5+6t\right)+C$

3.) $-\frac{1}{4}\left(3-2x-4x^2\right)^2+C$

4.) $-\frac{1}{3x}+C$

5.) $-\frac{15}{8}(1-x^2)^{\frac{4}{3}}+C$

6.) $\frac{(x+5)^7}{7}+C$

7.) $\frac{(3x-8)^{11}}{33}+C$

Practice with Integration by Substitution Involving Trig

Evaluate the integral.

1. $\int \cos 6x dx$
2. $\int x \sin x^2 dx$
3. $\int \csc^2 \frac{x}{2} dx$
4. $\int \csc 2x \cot 2x dx$
5. $\int \sqrt{\cot x} \csc^2 x dx$
6. $\int \frac{\sin x}{\cos^2 x} dx$
7. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

Answers:

1. $\frac{1}{6} \sin 6x + C$
2. $\frac{-1}{2} \cos x^2 + C$
3. $-2 \cot \frac{x}{2} + C$
4. $\frac{-1}{2} \csc 2x + C$
5. $\frac{-2}{3} \cot^{3/2} x + C$
6. $\sec x + C$
7. $-2 \cos \sqrt{x} + C$

Trig Integrals HW

Integrate!

1.) $\int \cos 6x \, dx$

2.) $\int x \sin(x^2) \, dx$

3.) $\int \csc^2\left(\frac{x}{2}\right) dx$

4.) $\int \csc 2x \cot 2x \, dx$

5.) $\int \sqrt{\tan x} \sec^2 x \, dx$

6.) $\int \frac{\sin x}{\cos^2 x} dx$

7.) $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

Answers:

1.) $\frac{\sin 6x}{6} + C$

2.) $-\frac{\cos(x^2)}{2} + C$

3.) $-2\cot\left(\frac{x}{2}\right) + C$

4.) $-\frac{\csc 2x}{2} + C$

5.) $\frac{2}{3}\tan^{\frac{3}{2}} x + C$

6.) $\sec x + C$

7.) $-2\cos\sqrt{x} + C$

7.8 Numerical Integration

Learning Objectives

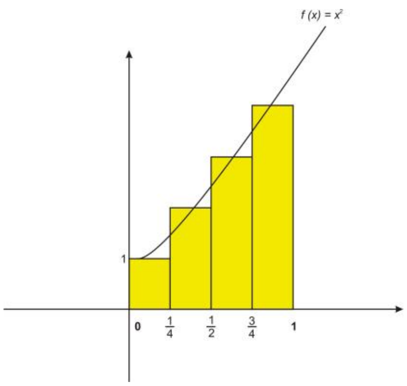
- Use the Trapezoidal Rule to solve problems
- Estimate errors for the Trapezoidal Rule
- Use Simpson's Rule to solve problems
- Estimate Errors for Simpson's Rule

Introduction

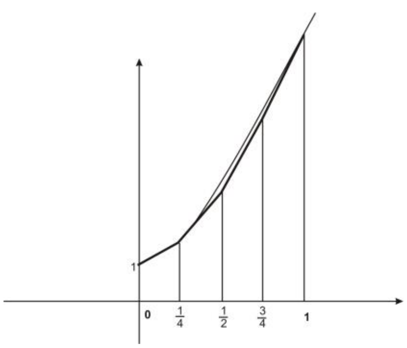
Recall that we used different ways to approximate the value of integrals. These included Riemann Sums using left and right endpoints, as well as midpoints for finding the length of each rectangular tile. In this lesson we will learn two other methods for approximating integrals. The first of these, the Trapezoidal Rule, uses areas of trapezoidal tiles to approximate the integral. The second method, Simpson's Rule, uses parabolas to make the approximation.

Trapezoidal Rule

Let's recall how we would use the midpoint rule with $n = 4$ rectangles to approximate the area under the graph of $f(x) = x^2 + 1$ from $x = 0$ to $x = 1$.



If instead of using the midpoint value within each sub-interval to find the length of the corresponding rectangle, we could have instead formed trapezoids by joining the maximum and minimum values of the function within each sub-interval:



The area of a trapezoid is $A = \frac{h(b_1+b_2)}{2}$, where b_1 and b_2 are the lengths of the parallel sides and h is the height. In our trapezoids the height is $\triangle x$ and b_1 and b_2 are the values of the function. Therefore in finding the areas of the trapezoids we actually average the left and right endpoints of each sub-interval. Therefore a typical trapezoid would have the area

$$A = \frac{\triangle x}{2}\left(f(x_{i-1}) + f(x_i)\right).$$

To approximate $\int_a^b f(x)dx$ with n of these trapezoids, we have

$$\int_a^b f(x)dx \approx \frac{1}{2}\left[\sum_{i-1}^{n} f(x_{i-1})\triangle x + \sum_{i-1}^{n} f(x_i)\triangle x\right]$$
$$= \frac{\triangle x}{2}\left[f(x_0)+f(x_1)+f(x_1)+f(x_2)+f(x_2)+\ldots+f(x_{n-1})f(x_n)\right]$$
$$= \frac{\triangle x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})f(x_n)\right], \triangle x = \frac{b-a}{n}.$$

Example 1:

Use the Trapezoidal Rule to approximate $\int_0^3 x^2 dx$ with $n = 6$.

Solution:

We find $\triangle x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$.

$$\int_0^3 x^2 dx \approx \frac{1}{4}\left[f(0)+2f(\tfrac{1}{2})+2f(1)+2f(\tfrac{3}{2})+2f(2)+2f(\tfrac{5}{2})+f(3)\right]$$
$$= \frac{1}{4}\left[0+(2\cdot\tfrac{1}{4})+(2\cdot 1)+(2\cdot\tfrac{9}{4})+(2\cdot 4)+(2\cdot\tfrac{25}{4})+9\right]$$
$$= \frac{1}{4}\left[\tfrac{73}{2}\right] = \frac{73}{8} = 9.125.$$

Of course, this estimate is not nearly as accurate as we would like. For functions such as $f(x) = x^2$, we can easily find an antiderivative with which we can apply the Fundamental Theorem that $\int_0^3 x^2 dx = \frac{x^3}{3}\Big]_0^3 = 9.$ But it is not always easy to find an antiderivative. Indeed, for many integrals it is impossible to find an antiderivative. Another issue concerns the questions about the accuracy of the approximation. In particular, how large should we take n so that the Trapezoidal Estimate for $\int_0^3 x^2 dx$ is accurate to within a given value, say 0.001? As with our Linear Approximations in the Lesson on Approximation Errors, we can state a method that ensures our approximation to be within a specified value.

Error Estimates for Simpson's Rule

We would like to have confidence in the approximations we make. Hence we can choose n to ensure that the errors are within acceptable boundaries. The following method illustrates how we can choose a sufficiently large n.

Suppose $|f''(x)| \leq k$ for $a \leq x \leq b$. Then the error estimate is given by

$$|Error_{Trapezoidal}| \leq \frac{k(b-a)^3}{12n^2}.$$

Example 2:

Find n so that the Trapezoidal Estimate for $\int_0^3 x^2 dx$ is accurate to 0.001.

Solution:

We need to find n such that $|Error_{Trapezoidal}| \leq 0.001$. We start by noting that $|f''(x)| = 2$ for $0 \leq x \leq 3$. Hence we can take $K = 2$ to find our error bound.

$$|Error_{Trapezoidal}| \leq \frac{2(3-0)^3}{12n^2} = \frac{54}{12n^2}.$$

We need to solve the following inequality for n:

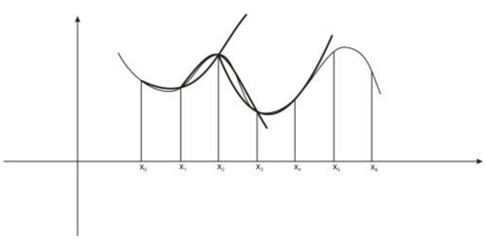
$$\frac{54}{12n^2} < 0.001,$$
$$n^2 > \frac{54}{12(0.001)},$$
$$n > \sqrt{\frac{54}{12(0.001)}} \approx 67.08.$$

Hence we must take $n = 68$ to achieve the desired accuracy.

From the last example, we see one of the weaknesses of the Trapezoidal Rule—it is not very accurate for functions where straight line segments (and trapezoid tiles) do not lead to a good estimate of area. It is reasonable to think that other methods of approximating curves might be more applicable for some functions. ***Simpson's Rule*** is a method that uses parabolas to approximate the curve.

Simpson's Rule:

As was true with the Trapezoidal Rule, we divide the interval $[a, b]$ into n sub-intervals of length $\triangle x = \frac{b-a}{n}$. We then construct parabolas through each group of three consecutive points on the graph. The graph below shows this process for the first three such parabolas for the case of $n = 6$ sub-intervals. You can see that every interval except the first and last contains two estimates, one too high and one too low, so the resulting estimate will be more accurate.



Using parabolas in this way produces the following estimate of the area from Simpson's Rule:

$$\int_a^b f(x)dx \approx \frac{\triangle x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

We note that it has a similar appearance to the Trapezoidal Rule. However, there is one distinction we need to note. The process of using three consecutive x_i to approximate parabolas will require that we assume that n must always be an even number.

Error Estimates for the Trapezoidal Rule

As with the Trapezoidal Rule, we have a formula that suggests how we can choose n to ensure that the errors are within acceptable boundaries. The following method illustrates how we can choose a sufficiently large n.

Suppose $|f^4(x)| \leq k$ for $a \leq x \leq b$. Then the error estimate is given by

$$|Error_{simpson}| \leq \frac{k(b-a)^5}{180n^4}.$$

Example 3:

a. Use Simpson's Rule to approximate $\int_1^4 \frac{1}{x} dx$ with $n = 6$.

Solution:

We find $\triangle x = \frac{b-a}{n} = \frac{4-1}{6} = \frac{1}{2}$.

$$\begin{aligned}\int_1^4 \frac{1}{x} dx &\approx \frac{1}{6}\left[f(1) + 4f(\tfrac{3}{2}) + 2f(2) + 4f(\tfrac{5}{2}) + 2f(3) + 4f(\tfrac{7}{2}) + f(4)\right] \\ &= \frac{1}{6}\left[1 + (4 \cdot \tfrac{2}{3}) + (2 \cdot \tfrac{1}{2}) + (4 \cdot \tfrac{2}{5}) + (2 \cdot \tfrac{1}{3}) + (4 \cdot \tfrac{2}{7}) + \tfrac{1}{4}\right] \\ &= \frac{1}{6}\left[\tfrac{3517}{420}\right] = 1.3956.\end{aligned}$$

This turns out to be a pretty good estimate, since we know that

$$\int_1^4 \frac{1}{x} dx = \ln x \Big]_1^4 = \ln(4) - \ln(1) = 1.3863.$$

Therefore the error is less than 0.01.

b. Find n so that the Simpson Rule Estimate for $\int_1^4 \frac{1}{x} dx$ is accurate to 0.001.

Solution:

We need to find n such that $|Error_{simpson}| \leq 0.001$. We start by noting that $|f^4(x)| = |\frac{24}{x^5}|$ for $1 \leq x \leq 4$. Hence we can take $K = 24$ to find our error bound:

$$|Error_{simpson}| \leq \frac{24(4-1)^5}{180n^4} = \frac{5832}{180n^4}.$$

Hence we need to solve the following inequality for n:

$$\frac{5832}{180n^4} < 0.001.$$

We find that

$$n^4 > \frac{5832}{180(0.001)},$$

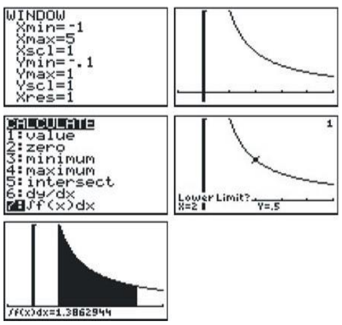
$$n > 4\sqrt{\frac{5832}{180(0.001)}} \approx 13.42.$$

Hence we must take $n = 14$to achieve the desired accuracy.

Technology Note: Estimating a Definite Integral with a TI-83/84 Calculator

We will estimate the value of $\int_1^4 \frac{1}{x} dx$.

1. Graph the function $f(x) = \frac{1}{x}$with the [WINDOW] setting shown below.
2. The graph is shown in the second screen.
3. Press **2nd [CALC]** and choose option **7** (see menu below)
4. When the fourth screen appears, press **[1] [ENTER]** then **[4] [ENTER]** to enter the lower and upper limits.
5. The final screen gives the estimate, which is accurate to 7 decimal places.

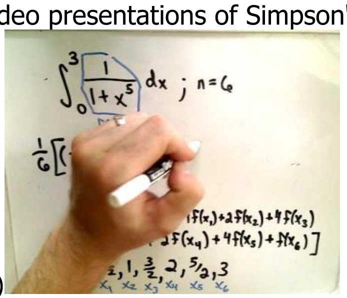


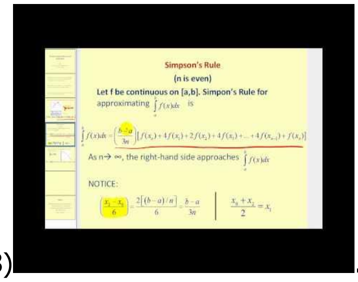
Lesson Summary

1. We used the Trapezoidal Rule to solve problems.
2. We estimated errors for the Trapezoidal Rule.
3. We used Simpson's Rule to solve problems.
4. We estimated Errors for Simpson's Rule.

Multimedia Links

For video presentations of Simpson's Rule **(21.0)**, see Simpson's Rule, Approximate Integration

(7:21) 

and Math Video Tutorials by James Sousa, Simpson's Rule of Numerical Integration (8:48)

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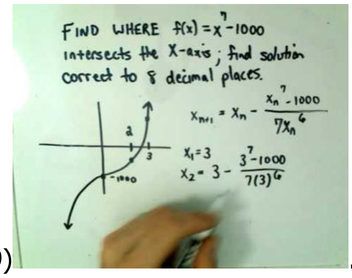
For a video presentation of Newton's Method **(21.0)**, see Newton's Method (7:29)

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Review Question

1. Use the Trapezoidal Rule to approximate $\int_0^1 x^2 e^{-x} dx$ with $n = 8.$
2. Use the Trapezoidal Rule to approximate $\int_1^4 \ln\sqrt{x} dx$ with $n = 6.$
3. Use the Trapezoidal Rule to approximate $\int_0^1 \sqrt{1 + x^4} dx$ with $n = 4.$
4. Use the Trapezoidal Rule to approximate $\int_1^3 \frac{1}{x} dx$ with $n = 8.$
5. How large should you take n so that the Trapezoidal Estimate for $\int_1^3 \frac{1}{x} dx$ is accurate to within .001?
6. Use Simpson's Rule to approximate $\int_0^1 x^2 e^{-x} dx$ with $n = 8.$
7. Use Simpson's Rule to approximate $\int_1^4 \sqrt{x} \ln\, x dx$ with $n = 6.$
8. Use Simpson's Rule to approximate $\int_0^2 \frac{1}{\sqrt{x^4+1}} dx$ with $n = 6.$
9. Use Simpson's Rule to approximate $\int_0^1 \sqrt{1 + x^4} dx$ with $n = 4.$
10. How large should you take n so that the Simpson Estimate for $\int_0^2 e dx$ is accurate to within .00001?

Review Answers

1. $\int_0^1 x^2 e^{-x} dx \approx 0.16$
2. $\int_1^4 \ln\sqrt{x} dx \approx 1.26$
3. $\int_0^1 \sqrt{1 + x^4} dx \approx 1.10$
4. $\int_1^3 \frac{1}{x} dx \approx 1.10$
5. Take $n = 19$
6. $\int_0^1 x^2 e^{-x} dx \approx 0.16$
7. $\int_1^4 \sqrt{x}\ln\, x dx \approx 4.28$
8. $\int_0^2 \frac{1}{\sqrt{x^4+1}} dx \approx 1.36$
9. $\int_0^1 \sqrt{1 + x^4} dx \approx 1.09$
10. Take $n = 9$

7.9 Area Between Two Curves

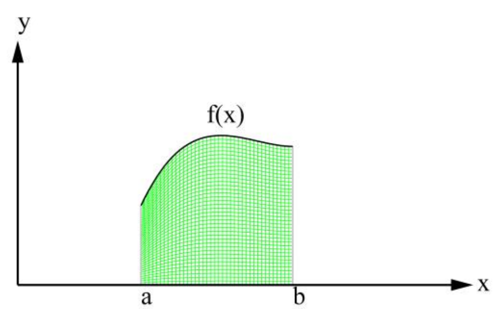
Learning Objectives

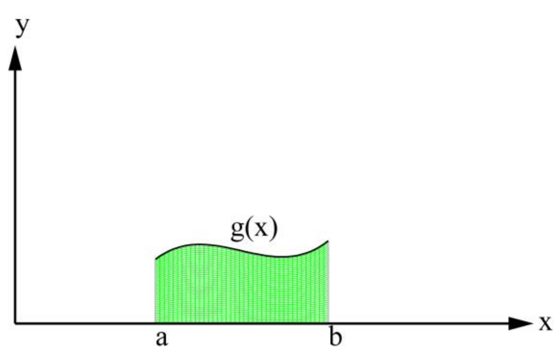
A student will be able to:

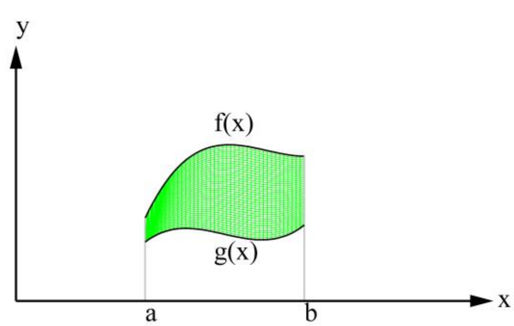
- Compute the area between two curves with respect to the $x-$and $y-$axes.

In the last chapter, we introduced the definite integral to find the area between a curve and the $x-$axis over an interval $[a, b]$. In this lesson, we will show how to calculate the area between two curves.

Consider the region bounded by the graphs $f$and $g$between $x = a$and $x = b$, as shown in the figures below. If the two graphs lie above the $x-$axis, we can interpret the area that is sandwiched between them as the area under the graph of $g$subtracted from the area under the graph f.







Therefore, as the graphs show, it makes sense to say that

[Area under f(Fig. 1a)] $-$[Area under g(Fig. 1b)] $=$[Area between f and g(Fig. 1c)],

$$\int_a^b f(x)dx - \int_a^b g(x) = \int_a^b [f(x)| - g(x)]dx.$$

This relation is valid as long as the two functions are continuous and the upper function $f(x) \geq g(x)$ on the interval $[a, b]$.

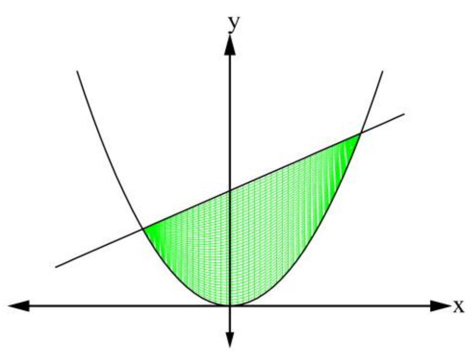
The Area Between Two Curves (*With respect to the $x-$axis*)

If f and g are two continuous functions on the interval $[a, b]$ and $f(x) \geq g(x)$ for all values of x in the interval, then the area of the region that is bounded by the two functions is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$

Example 1:

Find the area of the region enclosed between $y = x^2$ and $y = x + 6.$



Solution:

We first make a sketch of the region (Figure 2) and find the end points of the region. To do so, we simply equate the two functions,

$$x^2 = x + 6,$$

and then solve for x.

$$\begin{aligned} x^2 - x - 6 &= 0 \\ (x+2)(x-3) &= 0 \end{aligned}$$

from which we get $x = -2$ and $x = 3.$

So the upper and lower boundaries intersect at points $(-2, 4)$ and $(3, 9).$

As you can see from the graph, $x + 6 \geq x^2$ and hence $f(x) = x + 6$ and $g(x) = x^2$ in the interval $[-2, 3]$. Applying the area formula,

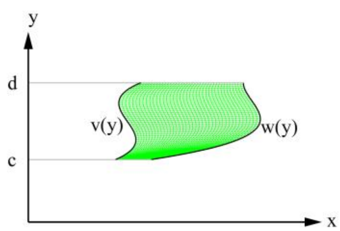
$$A = \int_a^b [f(x) - g(x)]dx$$
$$= \int_{-2}^{3} [(x+6) - (x^2)]dx.$$

Integrating,

$$A = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$
$$= \frac{125}{6}.$$

So the area between the two curves $f(x) = x + 6$ and $g(x) = x^2$ is $125/6.$

Sometimes it is possible to apply the area formula with respect to the $y-$coordinates instead of the $x-$coordinates. In this case, the equations of the boundaries will be written in such a way that y is expressed explicitly as a function of x (Figure 3).



The Area Between Two Curves (*With respect to the $y-$axis*)

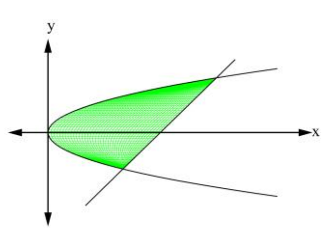
If w and v are two continuous functions on the interval $[c, d]$ and $w(y) \geq v(y)$ for all values of y in the interval, then the area of the region that is bounded by $x = v(y)$ on the left, $x = w(y)$ on the right, below by $y = c$, and above by $y = d$, is given by

$$A = \int_c^d [w(y) - v(y)]dy.$$

Example 2:

Find the area of the region enclosed by $x = y^2$ and $y = x - 6.$

Solution:



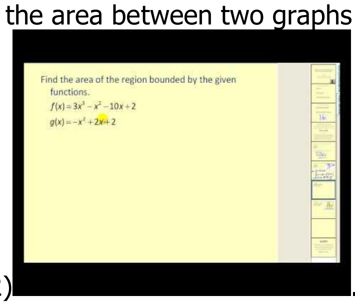
As you can see from Figure 4, the left boundary is $x = y^2$and the right boundary is $y = x - 6$.The region extends over the interval $-2 \le y \le 3$.However, we must express the equations in terms of y.We rewrite

$$x = y^2$$
$$x = y + 6$$

Thus

$$\begin{aligned} A &= \int_{-2}^{3} [y + 6 - y^2] dy \\ &= \left[\frac{y^2}{2} + 6y - \frac{y^3}{3} \right]_{-2}^{3} \\ &= \frac{125}{6}. \end{aligned}$$

Multimedia Links

For a video presentation of the area between two graphs **(14.0)(16.0)**, see Math Video Tutorials by James Sousa, Area Between Two Graphs (6:12)

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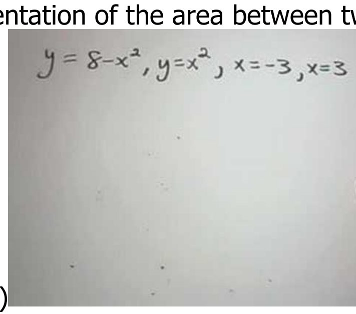
For an additional video presentation of the area between two curves **(14.0)(16.0)**, see Just Math Tutoring, Finding Areas Between Curves (9:50)

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Review Questions

In problems #1 - 7, sketch the region enclosed by the curves and find the area.

1. $y = x^2, y = \sqrt{x},$ on the interval $[0.25, 1]$
2. $y = 0, y = \cos 2x,$ on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
3. $y = |-1 + x| + 2, y = \frac{-1}{5}x + 7$
4. $y = \cos x, y = \sin x,$ [0, 2π]
5. $x = y^2, y = x - 2,$ integrate with respect to y
6. $y^2 - 4x = 4, 4x - y = 16$
7. $y = 8\cos x, y = sec^2 x, -\pi/3 \leq x \leq \pi/3$
8. Find the area enclosed by $x = y^3$ and $x = y.$
9. If the area enclosed by the two functions $y = k\cos x$ and $y = kx^2$ is 2, what is the value of k?
10. Find the horizontal line y = k that divides the region between $y = x^2$ and $y = 9$ into two equal areas.

Review Answers

1. $49/192$
2. $1/2$
3. 24
4. $4\sqrt{2}$
5. $9/2$
6. $30\frac{3}{8}$
7. $6\sqrt{3}$
8. $\frac{1}{2}$
9. $k \approx 1.83$
10. $y = \frac{9}{\sqrt[3]{4}}$

Area Between Two Curves Practice

Sketch the region bounded by the graphs of the functions and find the area of the region. You may use fnInt to calculate the area, but graph without the aide of your calculator!

1. $f(x) = x^2 - 4x,\ g(x) = 0$
2. $f(x) = x^2 + 2x + 1,\ g(x) = x + 1$
3. $y = 2x,\ y = 4 - 2x,\ y = 0$
4. $f(x) = x^2 - x,\ g(x) = 2(x+2)$
5. $y = x^3 - 2x + 1,\ y = -2x,\ x = 1$
6. $y = \sqrt{3x} + 1,\ g(x) = x + 1$
7. $y = x^2 - 4x + 3,\ y = 3 + 4x - x^2$
8. $f(y) = y^2,\ g(y) = y + 2$
9. $x = y^2 + 2,\ x = 0,\ y = -1,\ y = 2$
10. $y = \frac{4}{x},\ y = x,\ x = 1,\ x = 4$

Answers:

1. $\frac{32}{3}$
2. $\frac{1}{6}$
3. 2
4. $\frac{125}{6}$
5. 2
6. $\frac{3}{2}$
7. $\frac{64}{3}$
8. $\frac{9}{2}$
9. 9
10. $\frac{9}{2}$

Area Between 2 Curves HW

Find the area between the curves.

1.) $x = y^2; x = y + 2$

2.) $y^2 - 4x = 4; 4x - y = 16$

3.) $x = y^3; x = y$

4.) $y = x^2 - x; y = 2x + 4$

5.) $y = x^3 - 2x + 1; y = -2x; x = 1$

Answers:

1.) $\int_{-1}^{2} (y+2) - (y^2)\, dy = \frac{9}{2}$

2.) $\int_{-4}^{5} \left(\frac{16+y}{4}\right) - \left(\frac{y^2-4}{4}\right) dy = \frac{243}{8}$

3.) $\int_{-1}^{0} y^3 - y\, dy + \int_{0}^{1} y - y^3\, dy = \frac{1}{2}$

4.) $\int_{-1}^{4} (2x+4) - (x^2 - x)\, dx = \frac{125}{6}$

5.) $\int_{-1}^{1} (x^3 - 2x + 1) - (-2x)\, dx = 2$