8.1 Volumes

Learning Objectives

- Learn the basic concepts of volume and how to compute it with a given cross-section
- Learn how to compute volume by the disk method
- Learn how to compute volume by the washer method
- Learn how to compute volume by cylindrical shells

In this section, we will use definite integrals to find volumes of different solids.

The Volume Formula

A circular cylinder can be generated by translating a circular disk along a line that is perpendicular to the disk (Figure 5). In other words, the cylinder can be generated by moving the cross-sectional area \( A \) (the disk) through a distance \( h \). The resulting volume is called the volume of solid and it is defined to be

\[
V = Ah.
\]

The volume of solid does not necessarily have to be circular. It can take any arbitrary shape. One useful way to find the volume is by a technique called “slicing.” To explain the idea, suppose a solid \( S \) is positioned on the \( x \)-axis and extends from points \( x = a \) to \( x = b \) (Figure 6). Let \( A(x) \) be the cross-sectional area of the solid at some arbitrary point \( x \). Just like we did in calculating the definite integral in the previous chapter, divide the interval \( [a, b] \) into \( n \) sub-intervals and with widths

\[
\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n.
\]

Eventually, we get planes that cut the solid into \( n \) slices

\[
S_1, S_2, S_3, \ldots, S_n.
\]
Take one slice, $S_k$. We can approximate slice $S_k$ to be a rectangular solid with thickness $\triangle x_k$ and cross-sectional area $A(x_k)$. Thus the volume $V_k$ of the slice is approximately

$$V_k \approx A(x_k) \triangle x_k.$$ 

Therefore the volume $V$ of the entire solid is approximately

$$V = V_1 + V_2 + \ldots + V_n \approx \sum_{k=1}^{n} A(x_k) \triangle x_k.$$ 

If we use the same argument to derive a formula to calculate the area under the curve, let us increase the number of slices in such a way that $\triangle x_k \to 0$. In this case, the slices become thinner and thinner and, as a result, our approximation will get better and better. That is,

$$V = \lim_{\triangle x \to 0} \sum_{k=1}^{n} A(x_k) \triangle x_k.$$ 

Notice that the right-hand side is just the definition of the definite integral. Thus

$$V = \lim_{\triangle x \to 0} \sum_{k=1}^{n} A(x_k) \triangle x_k = \int_{a}^{b} A(x)dx.$$ 

**The Volume Formula** (*Cross-section perpendicular to the $x$-axis*)

Let $S$ be a solid bounded by two parallel planes perpendicular to the $x$-axis at $x = a$ and $x = b$. If each of the cross-sectional areas in $[a, b]$ are perpendicular to the $x$-axis, then the volume of the solid is given by

$$V = \int_{a}^{b} A(x)dx.$$ 

where $A(x)$ is the area of a cross section at the value of $x$ on the $x$-axis.

**The Volume Formula** (*Cross-section perpendicular to the $y$-axis*)

Let $S$ be a solid bounded by two parallel planes perpendicular to the $y$-axis at $y = c$ and $y = d$. If each of the cross-sectional areas in $[c, d]$ are perpendicular to the $y$-axis, then the volume of the solid is given by

$$V = \int_{c}^{d} A(y)dy.$$ 

where $A(y)$ is the area of a cross section at the value of $y$ on the $y$-axis.

**Example 1:**

Derive a formula for the volume of a pyramid whose base is a square of sides $a$ and whose height (altitude) is $h$. 

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2
Solution:

Let the $y$-axis pass through the apex of the pyramid, as shown in Figure (7a). At any point $y$ in the interval $[0, h]$ the cross-sectional area is a square. If $b$ is the length of the sides of any arbitrary square, then, by similar triangles (Figure 7b),

$$\frac{1}{2} b = \frac{h - y}{h},$$

$$b = \frac{a}{h} (h - y).$$

Since the cross-sectional area at $y$ is $A(y) = b^2$,

$$A(y) = b^2 = \frac{a^2}{h^2} (h - y)^2.$$

Using the volume formula,

$$V = \int_0^h A(y) dy$$

$$= \int_0^h \frac{a^2}{h^2} (h - y)^2 dy$$

$$= \frac{a^2}{h^2} \int_0^h (h - y)^2 dy.$$

Using $u$-substitution to integrate, we eventually get
Therefore the volume of the pyramid is \( V = \frac{1}{3}a^2h \), which agrees with the standard formula.

\[
V = \frac{a^2}{n^2} \left[ \frac{1}{3} (h - y)^3 \right]_0^h \\
= \frac{1}{3}a^2h.
\]

Volumes of Solids of Revolution

The Method of Disks

Suppose a function \( f \) is continuous and non-negative on the interval \([a, b]\) and suppose that \( R \) is the region between the curve \( f \) and the \( x \)-axis (Figure 8a). If this region is revolved about the \( x \)-axis, it will generate a solid that will have circular cross-sections (Figure 8b) with radii of \( f(x) \) at each \( x \). Each cross-sectional area can be calculated by

\[
A(x) = \pi [f(x)]^2.
\]

Since the volume is defined as

\[
V = \int_a^b A(x) \, dx,
\]
the volume of the solid is

\[ V = \int_{a}^{b} \pi [f(x)]^2 \, dx. \]

**Volumes by the Method of Disks** (*revolution about the \(x\)-axis*)

\[ V = \int_{a}^{b} \pi [f(x)]^2 \, dx. \]

Because the shapes of the cross-sections are circular or look like the shapes of disks, the application of this method is commonly known as the **method of disks**.

**Example 2**

Calculate the volume of the solid that is obtained when the region under the curve \(\sqrt{x}\) is revolved about the \(x\)-axis over the interval \([1, 7]\).

**Solution:**

As Figures 9a and 9b show, the volume is

\[
V = \int_{a}^{b} \pi [f(x)]^2 \, dx \\
= \int_{1}^{7} \pi [\sqrt{x}]^2 \, dx \\
= \pi \left[ \frac{x^2}{2} \right]_{1}^{7} \\
= 24\pi.
\]
Example 3:

Derive a formula for the volume of the sphere with radius $r$.

**Solution:**

One way to find the formula is to use the disk method. From your algebra, a circle of radius $r$ and center at the origin is given by the formula

$$x^2 + y^2 = r^2$$

If we revolve the circle about the $x$–axis, we will get a sphere. Using the disk method, we will obtain a formula for the volume. From the equation of the circle above, we solve for $y$:

$$f(x) = y = \sqrt{r^2 - x^2},$$

thus

$$V = \int_{a}^{b} \pi [f(x)]^2 dx$$

$$= \int_{-r}^{r} \pi \left[ \sqrt{r^2 - x^2} \right]^2 dx$$

$$= \pi \left[ r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

$$= \frac{4}{3} \pi r^3.$$

This is the standard formula for the volume of the sphere.

**The Method of Washers**

To generalize our results, if $f$ and $g$ are non-negative and continuous functions and

$$f(x) \geq g(x)$$

for

$$a \leq x \leq b,$$

Then let $R$ be the region enclosed by the two graphs and bounded by $x = a$ and $x = b$. When this region is revolved about the $x$–axis, it will generate washer-like cross-sections (Figures 10a and 10b). In this case, we will have two radii: an inner radius $g(x)$ and an outer radius $f(x)$. The volume can be given by:

$$V(x) = \int_{a}^{b} \pi \left( [f(x)]^2 - [g(x)]^2 \right) dx.$$
Volumes by the Method of Washers (*revolution about the \(x\)-axis*)

\[
V(x) = \int_a^b \pi \left( [f(x)]^2 - [g(x)]^2 \right) \, dx.
\]

**Example 4:**

Find the volume generated when the region between the graphs \(f(x) = x^2 + 1\) and \(g(x) = x\) over the interval \([0, 3]\) is revolved about the \(x\)-axis.

**Solution:**
From the formula above,

\[ V(x) = \int_a^b \pi \left( [f(x)]^2 - [g(x)]^2 \right) dx \]

\[ = \int_0^3 \pi \left( (x^2 + 1)^2 - (x^2)^2 \right) dx \]

\[ = \int_0^3 \pi (x^4 + x^2 + 1) dx \]

\[ = \frac{303\pi}{5}. \]

The methods of disks and washers can also be used if the region is revolved about the \( y \)-axis. The analogous formulas can be easily deduced from the above formulas or from the volumes of solids generated.

**Disks:**

\[ V = \int_c^d \pi [u(y)]^2 dy. \]

**Washers:**

\[ V = \int_c^d \pi \left( [w(y)]^2 - [v(y)]^2 \right) dy. \]

**Example 5:**

What is the volume of the solid generated when the region enclosed by \( y = \sqrt{x}, y = 3 \), and \( x = 0 \) is revolved about the \( y \)-axis?

**Solution:**

Since the solid generated is revolved about the \( y \)-axis (Figure 12), we must rewrite \( y = \sqrt{x} \) as \( x = y^2 \).

Thus \( u(y) = y^2 \) The volume is
\[ V = \int_{c}^{d} \pi [v(y)]^2 dy \]
\[ = \int_{0}^{3} \pi [s]^2 dy \]
\[ = \int_{0}^{3} \pi y^4 dx \]
\[ = \pi \left[ \frac{y^5}{5} \right]_{0}^{3} \]
\[ = \pi \left[ \frac{3^5}{5} - 0 \right] \]
\[ = \frac{243\pi}{5} . \]
Volume By Cylindrical Shells

The method of computing volumes so far depended upon computing the cross-sectional area of the solid and then integrating it across the solid. What happens when the cross-sectional area cannot be found or the integration is too difficult to solve? Here is where the shell method comes along.

To show how difficult it sometimes is to use the disk or the washer methods to compute volumes, consider the region enclosed by the function \( f(x) = x - x^2 \). Let us revolve it about the line \( x = -1 \) to generate the shape of a doughnut-shaped cake. What is the volume of this solid?

If we wish to integrate with respect to the \( y \)-axis, we have to solve for \( x \) in terms of \( y \). That would not be easy (try it!). An easier way is to integrate with respect to the \( x \)-axis by using the shell method. Here is how: A cylindrical shell is a solid enclosed by two concentric cylinders. If the inner radius is \( r_1 \) and the outer one is \( r_2 \) with both of height \( h \), then the volume is (Figure 14)
Notice however that $(r_2 - r_1)$ is the thickness of the shell and $\frac{1}{2}(r_2 + r_1)$ is the average radius of the shell.

Thus

$$V = 2\pi \cdot \text{[average radius]} \cdot \text{[height]} \cdot \text{[thickness]}.$$}

Replacing the average radius with a single variable $r$ and using $h$ for the height, we have

$$V = 2\pi \cdot r \cdot h \cdot \text{[thickness]}.$$}

In general the shell’s thickness will be $dx$ or $dy$ depending on the axis of revolution. This discussion leads to the following formulas for rotation about an axis. We will then use this formula to compute the volume $V$ of the solid of revolution that is generated by revolving the region about the $x-$axis.

**Volume By Cylindrical Shell about the $y-$Axis**

Suppose $f$ is a continuous function in the interval $[a, b]$ and the region $R$ is bounded above by $y = f(x)$ and below by the $x-$axis, and on the sides by the lines $x = a$ and $x = b.$ If $R$ is rotated around the $y-$axis, then the cylinders are vertical, with $r = x$ and $h = f(x).$ The volume of the solid is given by

$$V = \int_a^b 2\pi r h dx = \int_a^b 2\pi x f(x) dx.$$}

**Volume By Cylindrical Shell about the $x-$Axis**

Equivalently, if the volume is generated by revolving the same region about the $x-$axis, then the cylinders are horizontal with

$$v = \int_c^d 2\pi r h dy,$$

where $c = f^{-1}(a)$ and $d = f^{-1}(b).$ The values of $r$ and $h$ are determined in context, as you will see in Example 6.

Note: Example 7 shows what to do when the rotation is not about an axis.

**Example 6:**

A solid figure is created by rotating the region $R$ (Figure 15) around the $x-$axis. $R$ is bounded by the curve $y = x^2$ and the lines $x = 0$ and $x = 2.$ Use the shell method to compute the volume of the solid.
Solution:

From Figure 15 we can identify the limits of integration: \( y \) runs from 0 to 4. A horizontal strip of this region would generate a cylinder with height \( 2 - \sqrt{y} \) and radius \( y \). Thus the volume of the solid will be

\[
V = \int_0^4 2\pi rh
dy
\]

\[
= \int_0^4 2\pi y(2 - \sqrt{y})dy
\]

\[
= 2\pi \int_0^4 (2y - y^{3/2})dy
\]

\[
= 2\pi \left[ y^2 - \frac{2}{5}y^{5/2} \right]_0^4
\]

\[
= \frac{32\pi}{5}.
\]

Note: The alert reader will have noticed that this example could be worked with a simpler integral using disks. However, the following example can only be solved with shells.

Example 7:

Find the volume of the solid generated by revolving the region bounded by \( y = x^3 + \frac{1}{2}x + \frac{1}{4}y = \frac{1}{4} \), and \( x = 1 \), about \( x = 3 \).

Solution:

As you can see, the equation \( y = x^3 + \frac{1}{2}x + \frac{1}{4} \) cannot be easily solved for \( x \) and therefore it will be necessary to solve the problem by the shell method. We are revolving the region about a line parallel to the \( y \)-axis and thus integrate with respect to \( x \). Our formula is

\[
V = \int_a^b 2\pi rh
dx.
\]

In this case, the radius is \( 3 - x \) and the height is \( x^3 + \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} \). Substituting,
\[
V = 2\pi \int_{0}^{1} (3 - x) \left( x^3 + \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} \right) \, dx \\
= 2\pi \int_{0}^{1} \left( -x^4 + 3x^3 - \frac{1}{2}x^2 + \frac{3}{2}x \right) \, dx \\
= 2\pi \left[ -\frac{1}{5}x^5 + \frac{3}{4}x^4 - \frac{1}{6}x^3 + \frac{3}{4}x^2 \right]_{0}^{1} \\
= 2\pi \left[ -\frac{1}{5} + \frac{3}{4} - \frac{1}{6} + \frac{3}{4} \right] \\
= 2\pi \left[ \frac{17}{15} \right] \\
= \frac{34\pi}{15}.
\]

**Multimedia Links**

The following applet allows you to try out solids of revolution about the x-axis for any two functions. You can try inputting the examples above to test it out, and then experiment with new functions and changing the bounds. [Volumes of Revolution Applet](#). In the following video the narrator walks through the steps of setting up a volume integration problem:

![Volume Integration Video](image)


Sometimes the same volume problem can be solved in two different ways (14.0)(16.0). In these two videos, the narrator first finds a volume using shells [Khan Academy Solid of Revolution (Part 5)](#) (9:29), and then he does the same volume problem using disks. [Khan Academy Solid of Revolution (Part 6)](#)

![Volume Integration Video](image)

(9:18)

Together these videos show how both methods can be used to solve the same problem (though it's not always done this easily!).
Review Questions

In problems #1-4, find the volume of the solid generated by revolving the region bounded by the curves about the x-axis.

1. \( y = \sqrt{9-x^2}, y = 0 \)
2. \( y = 3 + x, y = 1 + x^2 \)
3. \( y = \sec x, y = \sqrt{2}, -\pi/4 \leq x \leq \pi/4 \)
4. \( y = 1, y = x, x = 0 \)

In problems #5-8, find the volume of the solid generated by revolving the region bounded by the curves about the y-axis.

5. \( y = x^3, x = 0, y = 1 \)
6. \( x = y^2, y = x - 2 \)
7. \( x = \csc y, y = \pi/4, y = 3\pi/4, x = 0 \)
8. \( y = 0, y = \sqrt{x}, x = 4 \)

In problems #9–12, use cylindrical shells to find the volume generated when the region bounded by the curves is revolved about the axis indicated.

9. \( y = \frac{1}{2}, y = 0, x = 1, x = 3 \) about the y-axis
10. \( y = x^2, x = 1, y = 0 \) about the x-axis
11. \( y = 2x - 1, y = -2x + 3, x = 2 \) about the y-axis
12. \( y^2 = x, y = 1, x = 0 \) about the x-axis.
13. Use the cylindrical shells method to find the volume generated when the region is bounded by \( y = x^3, y = 1, x = 0 \) is revolved about the line \( y = 1 \).

Review Answers

1. \( \frac{36\pi}{11} \)
2. \( \frac{3}{2} \)
3. \( \frac{\pi}{2} - 2\pi \)
4. \( \frac{3\pi}{2} \)
5. \( \frac{7\pi}{5} \)
6. \( \frac{2\pi}{5} \)
7. \( \frac{2\pi}{12} \)
8. \( \frac{5}{8} \)
9. \( 4\pi \)
10. \( \frac{\pi}{2} \)
11. \( \frac{120\pi}{3} \)
12. \( \frac{\pi}{2} \)
13. \( \frac{\pi}{14} \)
Practice on Disk/Washer Method

For #1 – 4, find the volume of the solid formed by revolving the region bounded by the graph(s) of the equation(s) about the x-axis.

1. \( y = 4 - x^2, y = 0 \)
2. \( y = -x + 1, y = 0, x = 0 \)
3. \( y = \frac{1}{x} - \frac{1}{2}, y = -\frac{1}{2} x + 1 \)
4. \( y = x^2, y = 4x - x^2 \)

For #5 – 8, find the volume of the solid formed by revolving the region bounded by the graph(s) of the equation(s) about the y-axis.

5. \( y = \sqrt{16 - x^2}, y = 0, 0 \leq x \leq 4 \)
6. \( x = 1 - \frac{1}{2} y, x = 0, y = 0 \)
7. \( y = \sqrt{x}, y = 0, x = 4 \)
8. \( y = \sqrt{4 - x}, y = \sqrt{x}, y = 0 \)

9. The line segment from \((0,0)\) to \((6,3)\) is revolved about the x-axis to form a cone. What is the volume of the cone?

10. Use the Disk Method to verify that the volume of a sphere of radius \( r \) is \( \frac{4}{3} \pi r^3 \).

11. The upper half of the ellipse \( 9x^2 + 16y^2 = 144 \) is revolved about the x-axis to form a prolate spheroid (shaped like a football). Find the volume of the spheroid.

Answers:

1. \( \frac{512\pi}{15} \)
2. \( \frac{\pi}{3} \)
3. .083
4. \( \frac{32\pi}{3} \)
5. \( \frac{128\pi}{3} \)
6. \( \frac{2\pi}{3} \)
7. \( \frac{128\pi}{5} \)
8. 11.488
9. \( 18\pi \)
10. \( V = \pi \int_{-r}^{r} \left( \sqrt{r^2 - x^2} \right)^2 dx = \frac{4}{3} \pi r^3 \)
11. \( 48\pi \)
Disc Method HW

Find the volume of the solid created by revolving the given region around the given axis.

1.) \( y = \sqrt{4 - x^2} \) in Quadrant I around the x-axis

\[
V = \pi \int_a^b r^2 \, dr
\]

2.) \( y = x^2, x = 0, x = 1 \) around the x-axis

3.) \( y = \sqrt{x}, x = 1, x = 4 \) around the x-axis

4.) \( y = 4 - x^2, y = 0 \) around the x-axis

5.) \( x = y^2, x = 0, y = 1 \) around the y-axis

6.) \( x = -y^2 + 4y, x = 0, y = 1 \) around the y-axis

Answers:

1.) \( \frac{16\pi}{3} \)

2.) \( \frac{\pi}{5} \)

3.) \( \frac{15\pi}{2} \)

4.) \( \frac{512\pi}{15} \)

5.) \( \frac{\pi}{4} \)

6.) \( \frac{153\pi}{5} \)
Shell Method Practice

For #1 – 5, use the shell method to find the volume of the figure formed when the region bounded by the graphs of the given equations is rotated around the indicated axis.

1. \(y = x^2, \ y = 0, \ x = 2\), around the y-axis

2. \(y = x^2, \ y = 4x - x^2\), around the y-axis

3. \(y = 2 - x, \ y = 0, \ x = 4\), around the x-axis

4. \(y = \sqrt{x}, \ y = 0, \ x = 4\), around the x-axis

5. \(y = \frac{1}{x}, \ x = 1, \ x = 2, \ y = 0\) around the y-axis

6. If you were asked to do #5 but the region was being revolved around the x-axis, the shell method would not be the best choice. Do you see why? Please find the volume of the figure formed from the revolution around the x-axis of the region described in #5 using another method. Also, give the set-up for if the shell method was used.

For # 7 – 10, setup, but DO NOT EVALUATE.

7. Find the volume that results when the region bounded by \(y = 16 - x^2\) and \(y = 16 - 4x\) is rotated about the x-axis. Use the washer method.

8. Repeat #7, but revolve around the y-axis and use cylindrical shells.

9. Repeat #7, but revolve around the x-axis and use cylindrical shells.

10. Repeat #7, but revolve around the y-axis and use washers.

Answers:

1. \(8\pi\)  
2. \(\frac{16\pi}{3}\)  
3. \(\frac{8\pi}{3}\)  
4. \(8\pi\)  
5. \(2\pi\)

6. Use disk method and get \(\frac{\pi}{2}\); shell mthd set-up: \(2\pi \int_0^5 ydy + 2\pi \int_0^1 y\left(\frac{1}{y} - 1\right)dy\)

7. \(\pi \int_0^4 \left[(16 - x^2)^2 - (16 - 4x)^2\right]dx\)  
8. \(2\pi \int_0^4 (4x^2 - x^3)dx\)

9. \(2\pi \int_0^2 y\left(\sqrt{16 - y} - 4 + \frac{y}{4}\right)dy\)  
10. \(\pi \int_0^1 \left[(16 - y) - \left(\frac{16 - y}{4}\right)^2\right]dy\)
8.2 The Length of a Plane Curve

Learning Objectives

A student will be able to:

- Learn how to find the length of a plane curve for a given function.

In this section, we will consider the problem of finding the length of a plane curve. Formulas for finding the arcs of circles appeared in early historical records and they were known to many civilizations. However, very little was known about finding the lengths of general curves, such as the length of the curve \( y = x^2 \) in the interval \([0, 2]\), until the discovery of calculus in the seventeenth century.

In calculus, we define an arc length as the length of a plane curve \( y = f(x) \) over an interval \([a, b]\) (Figure 17). When the curve \( f(x) \) has a continuous first derivative \( f'(x) \) on \([a, b]\), we say that \( f \) is a smooth function (or smooth curve) on \([a, b]\).

The Arc Length Problem

If \( y = f(x) \) is a smooth curve on the interval \([a, b]\), then the arc length \( L \) of this curve is defined as

\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx.
\]

Example 1:

Find the arc length of the curve \( y = \frac{x^3}{2} \) on \([1, 3]\) (Figure 18).
**Solution:**

Since \( y = x^3 / 2 \).

\[
\frac{dy}{dx} = \frac{3}{2}x^{1/2}.
\]

Using the formula above, we get

\[
\int_a^b \sqrt{1 + \left[ f'(x) \right]^2} \, dx = \int_1^3 \sqrt{1 + \left[ \frac{3}{2}x^{1/2} \right]^2} \, dx
\]

\[
= \int_1^3 \sqrt{1 + \frac{9}{4}x} \, dx.
\]

Using \( u - \)substitution by letting \( u = 1 + \frac{9}{4}x \), then \( du = \frac{9}{4} \, dx \). Substituting, and remembering to change the limits of integration,

\[
L = \frac{4}{9} \int_{1/4}^{31/4} \sqrt{u} \, du
\]

\[
= \frac{4}{27} \left[ \sqrt{u} \right]_{1/4}^{31/4}
\]

\[
\approx 4.65.
\]

**Multimedia Links**

The formula you just used to find the length of a curve was derived by using line segments to approximate the curve. The derivation of that formula can be found at [Wikipedia Entry on Arc Length](https://en.wikipedia.org/wiki/Arc_length). In the following applet you can explore this further. Experiment with various curves and change the number of segments to see how changing the number of segments is related to approximating the arc length. [Arc Length Applet](https://www.mathsisfun.com/calculus/arc-length.html).

For video presentations showing how to obtain the arc length using parametric curves (16.0), see [Just Math Tutoring](https://www.youtube.com/watch?v=example_video_id), [Arc Length Using Parametric Curves, Example 1 (8:17)](https://www.youtube.com/watch?v=example_video_id) and [Just Math Tutoring, Arc Length Using Parametric Curves, Example 2 (7:27)](https://www.youtube.com/watch?v=example_video_id).
Review Questions

1. Find the arc length of the curve \( y = \frac{(x^2 + 2)^{3/2}}{3} \) on \([0, 3]\).

2. Find the arc length of the curve \( x = \frac{1}{6}y^3 + \frac{1}{2} \) on \( y \in [1, 2] \).

3. Integrate \( x = \int \frac{1}{\sqrt{\sec^4 t - 1}} dt \), \( -\frac{\pi}{4} \leq y \leq \frac{\pi}{4} \).

4. Find the length of the curve shown in the figure below. The shape of the graph is called the *asteroid* because it looks like a star. The equation of its graph is \( x^{2/3} + y^{2/3} = 1 \).

5. The figure below shows a suspension bridge. The cable has the shape of a parabola with equation \( kx^2 = y \). The suspension bridge has a total length of \( 2S \) and the height of the cable is \( h \) at each end. Show that the total length of the cable is

\[
L = 2 \int_0^S \sqrt{1 + \frac{4h^2}{S^4} x^2} \, dx.
\]

Review Answers

1. \( \frac{11}{12} \)
2. \( \frac{17}{12} \)
3. \( 2 \)
4. \( 6 \)
5. \( . \)

8.3 Area of a Surface of Revolution

Learning Objectives

A student will be able to:

- Learn how to find the area of a surface that is generated by revolving a curve about an axis or a line.

In this section we will deal with the problem of finding the area of a surface that is generated by revolving a curve about an axis or a line. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter
Area of a Surface of Revolution

If \( f \) is a smooth and non-negative function in the interval \([a, b]\), then the surface area \( S \) generated by revolving the curve \( y = f(x) \) between \( x = a \) and \( x = b \) about the \( x \)-axis is defined by

\[
S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.
\]

Equivalently, if the surface is generated by revolving the curve about the \( y \)-axis between \( y = c \) and \( y = d \), then

\[
S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} \, dy = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy.
\]

**Example 1:**

Find the surface area that is generated by revolving \( y = x^3 \) on \([0, 2]\) about the \( x \)-axis (Figure 21).

**Solution:**
The surface area \( S \) is

\[
S = \int_a^b 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

\[
= \int_0^2 2\pi x^3 \sqrt{1 + (3x^2)^2} \, dx
\]

\[
= 2\pi \int_0^2 x^3(1 + 9x^4)^{1/2} \, dx.
\]

Using \( u \)-substitution by letting \( u = 1 + 9x^4 \),

\[
S = 2\pi \int_1^{145} u^{1/2} \frac{du}{36}
\]

\[
= \frac{2\pi}{36} \left[ \frac{2}{3} u^{3/2} \right]_1^{145}
\]

\[
= \frac{2\pi}{36} \cdot \frac{2}{3} \left( 145^{3/2} - 1 \right)
\]

\[
\approx \frac{4\pi}{108} [1745]
\]

\[
\approx 203
\]

**Example 2:**

Find the area of the surface generated by revolving the graph of \( f(x) = x^2 \) on the interval \([0, \sqrt{3}]\) about the \( y \)-axis (Figure 22).

**Solution:**
Since the curve is revolved about the $y-$axis, we apply

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy.$$

So we write $y = x^2$ as $x = \sqrt{y}$. In addition, the interval on the $x-$axis $[0, \sqrt{3}]$ becomes $[0, 3]$. Thus

$$S = \int_0^3 2\pi \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2} \, dy.$$

Simplifying,

$$S = \pi \int_0^3 \sqrt{4y + 1} \, dy.$$

With the aid of $u-$substitution, let $u = 4y + 1$,

$$S = \frac{\pi}{4} \int_1^{13} u^{1/2} \, du = \frac{\pi}{6} \left(13^{3/2} - 1\right) = \frac{\pi}{6} [46.88 - 1] \approx 24$$

**Multimedia Links**

For video presentations of finding the surface area of revolution (16.0), see [Math Video Tutorials by James Sousa](https://www.youtube.com/watch?v=16.0).

**Surface Area of Revolution, Part 1 (9:47)**

**Sousa, Surface Area of Revolution, Part 2 (5:43)**
Review Questions

In problems #1-3 find the area of the surface generated by revolving the curve about the x-axis.

1. \( y = 3x, \ 0 \leq x \leq 1 \)
2. \( y = \sqrt{x}, \ 1 \leq x \leq 9 \)
3. \( y = \sqrt{4 - x^2}, \ -1 \leq x \leq 1 \)

In problems #4–6 find the area of the surface generated by revolving the curve about the y-axis.

4. \( x = 7y + 2, \ 0 \leq y \leq 3 \)
5. \( x = y^3, \ 0 \leq y \leq 8 \)
6. \( x = \sqrt{9 - y^2}, \ -2 \leq y \leq 2 \)
7. Show that the surface area of a sphere of radius \( r \) is \( 4\pi r^2 \).
8. Show that the lateral area \( S \) of a right circular cone of height \( h \) and base radius \( r \) is \( S = \pi r \sqrt{r^2 + h^2} \).

Review Answers

1. \(3\pi \sqrt{10}\)
2. \(\approx 112\)
3. \(8\pi\)
4. \(70\pi \sqrt{50}\)
5. \(\approx 823165.5\)
6. \(24\pi\)
7. ．
8. ．

8.4 Applications from Physics, Engineering, and Statistics

Learning Objectives

A student will be able to:

- Learn how to apply definite integrals to several applications from physics, engineering, and applied mathematics such as work, fluids statics, and probability.

In this section we will show how the definite integral can be used in different applications. Some of the concepts may sound new to the reader, but we will explain what you need to comprehend as we go along. We will take three applications: The concepts of work from physics, fluid statics from engineering, and the normal probability from statistics.

Work

Work in physics is defined as the product of the force and displacement. Force and displacement are vector quantities, which means they have a direction and a magnitude. For example, we say the compressor exerts a force of 200 Newtons (N) upward. The magnitude here is 200 N and the direction is upward. Lowering a book from an upper shelf to a lower one by a distance of 0.5 meters away from its initial position is another example of the vector nature of the displacement. Here, the magnitude is 0.5 m and the direction is downward, usually indicated by a minus sign, i.e., a displacement of \(-0.5\) m. The product of those two vector quantities (called the inner product, see Chapter 10) gives the work done by the force. Mathematically, we say

\[ W = Fd. \]
where $F$ is the force and $d$ is the displacement. If the force is measured in Newtons and distance is in meters, then work is measured in the units of energy which is in joules (J).

**Example 1:**

You push an empty grocery cart with a force of 44 N for a distance of 12 meters. How much work is done by you (the force)?

**Solution:**

Using the formula above,

$W = Fd$

$= (44)(12)$

$= 528$ J.

**Example 2:**

A librarian displaces a book from an upper shelf to a lower one. If the vertical distance between the two shelves is 0.5 meters and the weight of the book is 5 Newtons. How much work is done by the librarian?

**Solution:**

In order to be able to lift the book and move it to its new position, the librarian must exert a force that is at least equal to the weight of the book. In addition, since the displacement is a vector quantity, then the direction must be taken into account. So,

$d = -0.5$ meters.

Thus

$W = Fd$

$= (5)(-0.5)$

$= -2.5$ J.

Here we say that the work is negative since there is a loss of gravitational potential energy rather than a gain in energy. If the book is lifted to a higher shelf, then the work is positive, since there will be a gain in the gravitational potential energy.

**Example 3:**

A bucket has an empty weight of 23 N. It is filled with sand of weight 80 N and attached to a rope of weight 5.1 N/m. Then it is lifted from the floor at a constant rate to a height 32 meters above the floor. While in flight, the bucket leaks sand grains at a constant rate, and by the time it reaches the top no sand is left in the bucket. Find the work done:

1. by lifting the empty bucket;
2. by lifting the sand alone;
3. by lifting the rope alone;
4. by the lifting the bucket, the sand, and the rope together.

**Solution:**
1. **The empty bucket.** Since the bucket's weight is constant, the worker must exert a force that is equal to the weight of the empty bucket. Thus

\[ W = Fd \]
\[ = (23)(+32) \]
\[ = 736 \text{ J.} \]

2. **The sand alone.** The weight of the sand is decreasing at a constant rate from 80 N to 0 N over the 32 m lift. When the bucket is at \( x \) meters above the floor, the sand weighs

\[ F(x) = [\text{original weight of sand}] \frac{\text{proportion left at elevation } x}{32} \]
\[ = 80 \left( \frac{32-x}{32} \right) \]
\[ = 80 \left( 1 - \frac{x}{32} \right) \]
\[ = 80 - 2.5x \text{ N.} \]

The graph of \( F(x) = 80 - 2.5x \) represents the variation of the force with height \( x \) (Figure 23). The work done corresponds to computing the area under the force graph.

Thus the work done is

\[ W = \int_{0}^{32} (80 - 2.5x) \, dx \]
\[ = \left[ 80x - \frac{2.5}{2}x^2 \right]_{0}^{32} \]
\[ = 1280 \text{ J.} \]

3. **The rope alone.** Since the weight of the rope is \( 5.1 \text{ N/m} \) and the height is 32 meters, the total weight of the rope from the floor to a height of 32 meters is

\( (5.1)(32) = 163.2 \text{ N.} \)

But since the worker is constantly pulling the rope, the rope's length is decreasing at a constant rate and thus its weight is also decreasing as the bucket being lifted. So at \( x \) meters, the \( (32-x) \) meters of rope remain to be lifted of weight

\( F(x) = (5.1)(32-x) \text{ N.} \)

Thus the work done to lift the weight of the rope is
4. The bucket, the sand, and the rope together. Here we are asked to sum all the work done on the empty bucket, the sand, and the rope. Thus

\[ W_{\text{total}} = 736 + 1280 + 2611.2 = 4627.2 \text{ J.} \]

**Fluid Statics: Pressure**

You have probably studied that **pressure** is defined as the force per area

\[ P = \frac{F}{A}, \]

which has the units of Pascals (Pa) or Newtons per meter squared, \( \text{Pa} = \text{N/m}^2 \). In the study of fluids, such as water pressure on a dam or water pressure in the ocean at a depth \( h \), another equivalent formula can be used. It is called the **liquid pressure** \( P \) at depth \( h \):

\[ P = wh. \]

where \( w \) is the **weight density**, which is the weight of the column of water per unit volume. For example, if you are diving in a pool, the pressure of the water on your body can be measured by calculating the total weight that the column of water is exerting on you times your depth. Another way to express this formula, the weight density \( w \) is defined as

\[ w = \rho g, \]

where \( \rho \) is the density of the fluid and \( g \) is the acceleration due to gravity (which is \( g = 9.8 \text{m/sec}^2 \) on Earth). The pressure then can be written as

\[ P = wh = \rho gh. \]

**Example 4:**

What is the total pressure experienced by a diver in a swimming pool at a depth of 2 meters?

**Solution**

First we calculate the fluid pressure the water exerts on the diver at a depth of 2 meters:

\[ P = \rho gh. \]

The density of water is \( \rho = 1000 \text{ kg/m}^3 \), thus

\[ P = (1000)(9.8)(2) \]

\[ = 19600 \text{ Pa.} \]
The total pressure on the diver is the pressure due to the water plus the atmospheric pressure. If we assume that the diver is located at sea-level, then the atmospheric pressure at sea level is about \( 10^5 \text{ Pa} \). Thus the total pressure on the diver is

\[
P_{\text{total}} = P_{\text{water}} + P_{\text{atm}} \\
= 19600 + 10^5 \\
= 119600 \\
= 1.196 \times 10^5 \text{ Pa}.
\]

**Example 5:**

What is the fluid pressure (excluding the air pressure) and force on the top of a flat circular plate of radius 3 meters that is submerged horizontally at a depth of 10 meters?

**Solution:**

The density of water is \( \rho = 1000 \text{ kg/m}^3 \). Then

\[
P = \rho gh \\
= (1000)(9.8)(10) \\
= 98000 \text{ Pa}.
\]

Since the force is \( F = PA \), then

\[
F = PA \\
= P \cdot \pi r^2 \\
= (98000)(\pi)(3)^2 \\
= 2.77 \times 10^6 \text{ N}.
\]

As you can see, it is easy to calculate the fluid force on a horizontal surface because each point on the surface is at the same depth. The problem becomes a little complicated when we want to calculate the fluid force or pressure if the surface is vertical. In this situation, the pressure is not constant at every point because the depth is not constant at each point. To find the fluid force or pressure on a vertical surface we must use calculus.

**The Fluid Force on a Vertical Surface**

Suppose a flat surface is submerged vertically in a fluid of weight density \( \omega \) and the submerged portion of the surface extends from \( x = a \) to \( x = b \) along the vertical \( x \)-axis, whose positive direction is taken as downward. If \( L(x) \) is the width of the surface and \( h(x) \) is the depth of point \( x \), then the fluid force \( F \) is defined as

\[
F = \int_a^b \omega h(x)L(x)dx.
\]

**Example 6:**

A perfect example of a vertical surface is the face of a dam. We can picture it as a rectangle of a certain height and certain width. Let the height of the dam be 100 meters and of width of 300 meters. Find the total fluid force exerted on the face if the top of the dam is level with the water surface (Figure 24).
Solution:

Let $x$ be the depth of the water. At an arbitrary point $x$ on the dam, the width of the dam is $L(x) = 300$ m and the depth is $h(x) = x m$. The weight density of water is

$$w_{\text{water}} = \rho g = (1000)(9.8) = 9800 \text{ N/m}^2.$$

Using the fluid force formula above,

$$F = \int_{a}^{b} wh(x)L(x)dx = \int_{0}^{100} (9800)(x)(300)dx = 2.94 \times 10^6 \int_{0}^{100} x dx = 2.94 \times 10^6 \left[ \frac{x^2}{2} \right]_{0}^{100} = 1.47 \times 10^{10} \text{ N}.$$

Normal Probabilities

If you were told by the postal service that you will receive the package that you have been waiting for sometime tomorrow, what is the probability that you will receive it sometime between 3:00 PM and 5:00 PM if you know that the postal service’s hours of operations are between 7:00 AM to 6:00 PM?

If the hours of operations are between 7 AM to 6 PM, this means they operate for a total of 11 hours. The interval between 3 PM and 5 PM is 2 hours, and thus the probability that your package will arrive is

$$P = \frac{2 \text{ hours}}{11 \text{ hours}} = 0.182 = 18.2\%.$$  

So there is a probability of 18.2% that the postal service will deliver your package sometime between the hours of 3 PM and 5 PM (or during any 2 - hour interval). That is easy enough. However, mathematically, the situation is not that simple. The 11 - hour interval and the 2 - hour interval contain an infinite number of times. So how can one infinity
over another infinity produce a probability of $18.2\%$? To resolve this issue, we represent the total probability of the 11 - hour interval as a rectangle of area $1$ (Figure 25). Looking at the 2 - hour interval, we can see that it is equal to $\frac{2}{11}$ of the total rectangular area $1$. This is why it is convenient to represent probabilities as areas. But since areas can be defined by definite integrals, we can also define the probability associated with an interval $[a, b]$ by the definite integral

$$\int_{a}^{b} f(x) \, dx,$$

where $f(x)$ is called the probability density function (pdf). One of the most useful probability density functions is the normal curve or the Gaussian curve (and sometimes the bell curve) (Figure 26). This function enables us to describe an entire population based on statistical measurements taken from a small sample of the population. The only measurements needed are the mean $\mu$ and the standard deviation $\sigma$. Once those two numbers are known, we can easily find the normal curve by using the following formula.

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where the factor $1/(\sigma \sqrt{2\pi})$ is called the normalization constant. It is needed to make the probability over the entire space equal to 1. That is,

$$P(-\infty < x < \infty) = \int_{-\infty}^{+\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 1.$$

**Example 7:**

Suppose that boxes containing 100 tea bags have a mean weight of 10.2 ounces each and a standard deviation of 0.1 ounce.

1. What percentage of all the boxes is expected to weigh between 10 and 10.5 ounces?
2. What is the probability that a box weighs less than 10 ounces?
3. What is the probability that a box will weigh exactly 10 ounces?

**Solution:**

1. Using the normal probability density function,

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \]

Substituting for \( \mu = 10.2 \) and \( \sigma = 0.1 \) we get

\[ f(x) = \frac{1}{(0.1) \sqrt{2\pi}} e^{-\frac{(x-10.2)^2}{2(0.1)^2}}. \]

The percentage of all the tea boxes that are expected to weight between 10 and 10.5 ounces can be calculated as

\[ P(10 \leq x \leq 10.5) = \int_{10}^{10.5} \frac{1}{(0.1) \sqrt{2\pi}} e^{-\frac{(x-10.2)^2}{2(0.1)^2}} \, dx. \]

The integral of \( e^{x^2} \) does not have an elementary anti-derivative and therefore cannot be evaluated by standard techniques. However, we can use numerical techniques, such as The Simpson’s Rule or The Trapezoid Rule, to find an approximate (but very accurate) value. Using the programming feature of a scientific calculator or, mathematical software, we eventually get

\[ \int_{10}^{10.5} \frac{1}{(0.1) \sqrt{2\pi}} e^{-\frac{(x-10.2)^2}{2(0.1)^2}} \, dx \approx 0.976. \]

That is,

\[ P(10 \leq x \leq 10.2) \approx 97.6\%. \]

**Technology Note:** To make this computation with a graphing calculator of the TI-83/84 family, do the following:

- From the [DISTR] menu (Figure 27) select option 2, which puts the phrase "normalcdf" in the home screen. Add lower bound, upper bound, mean, standard deviation, separated by commas, close the parentheses, and press [ENTER]. The result is shown in Figure 28.
2. For the probability that a box weighs less than 10.2 ounces, we use the area under the curve to the left of \( x = 10.2 \). Since the value of \( f(9) \) is very small (less than a billionth),

\[
f(9) = \frac{1}{(0.1)\sqrt{2\pi}} e^{-\frac{(9-10.2)^2}{2(0.1)^2}} dx
\]

\[
= 1.35 \times 10^{-32},
\]

getting the area between 9 and 10 will yield a fairly good answer. Integrating numerically, we get

\[
P(9 \leq x \leq 10) = \int_{9}^{10} \frac{1}{(0.1)\sqrt{2\pi}} e^{-\frac{(x-10.2)^2}{2(0.1)^2}} dx
\]

\[
P(9 \leq x \leq 10.2) \approx 0.02275
\]

\[
= 2.28%,
\]

which says that we would expect 2.28% of the boxes to weigh less than 10 ounces.

3. Theoretically the probability here will be exactly zero because we will be integrating from 10 to 10, which is zero. However, since all scales have some error (call it \( \epsilon \)), practically we would find the probability that the weight falls between 10 - \( \epsilon \) and 10 + \( \epsilon \).

**Example 8:**

An Intelligence Quotient or IQ is a score derived from different standardized tests attempting to measure the level of intelligence of an adult human being. The average score of the test is 100 and the standard deviation is 15.

1. What is the percentage of the population that has a score between 85 and 115?
2. What percentage of the population has a score above 140?

**Solution:**

1. Using the normal probability density function,

\[
f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},
\]

and substituting \( \mu = 100 \) and \( \sigma = 15 \),

\[
f(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}}.
\]

The percentage of the population that has a score between 85 and 115 is

\[
P(85 \leq x \leq 115) = \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}} dx.
\]

Again, the integral of \( e^{-x^2} \) does not have an elementary anti-derivative and therefore cannot be evaluated. Using the programming feature of a scientific calculator or a mathematical computer software, we get

\[
\int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}} dx \approx 0.68.
\]
That is,

\[ P(85 \leq x \leq 115) \approx 68\%. \]

Which says that 68% of the population has an IQ score between 85 and 115.

2. To measure the probability that a person selected randomly will have an IQ score above 140,

\[ P(x \geq 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}} \, dx. \]

This integral is even more difficult to integrate since it is an improper integral. To avoid the messy work, we can argue that since it is extremely rare to meet someone with an IQ score of over 200, we can approximate the integral from 140 to 200, then

\[ P(x \geq 140) \approx \int_{140}^{200} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{2(15)^2}} \, dx. \]

Integrating numerically, we get

\[ P(x \geq 140) \approx 0.0039. \]

So the probability of selecting at random a person with an IQ score above 140 is 0.39%. That’s about one person in every 250 individuals!

**Multimedia Links**

For a video presentation of an application of integration involving consumer and producer surplus (14.0), see Math Video Tutorials by James Sousa, Consumer and Producer Surplus (10:22).

For video presentations of work and Hooke’s Law (14.0)(16.0), see Just Math Tutoring, Work and Hooke’s Law, Example 1 (5:00) and Just Math Tutoring, Work and Hooke’s Law, Example 2 (6:52).
Review Questions

1. A particle moves along the x-axis by a force \( F(x) = \frac{1}{x^2 + 1} \). If the particle has already moved a distance of 10 meters from the origin, what is the work done by the force?

2. A force of \( \cos \left( \frac{\pi x}{2} \right) \) acts on an object when it is x meters away from the origin. How much work is done by this force in moving the object from \( x = 1 \) to \( x = 5 \) meters?

3. In physics, if the force on an object varies with distance then work done by the force is defined as (see Example 5.15)
\[
W = \int_a^b F(r)dr.
\]
That is, the work done corresponds to computing the area under the force graph. For example, the strength of the gravitational field varies with the distance \( r \) from the Earth’s center. If a satellite of mass \( m \) is to be launched into space, then the force experienced by the satellite during and after launch is
\[
F(r) = G \frac{mM}{r^2},
\]
where \( M = 6 \times 10^{24} \text{ kg} \) is the mass of the Earth and \( G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \) is the Universal Gravitational Constant. If the mass of the satellite is 1000 kg and we wish to lift it to an altitude of 35,780 km above the Earth’s surface, how much work is needed to lift it? (Radius of Earth is 6370 km.)

4. Hook’s Law states that when a spring is stretched \( x \) units beyond its natural length it pulls back with a force \( F(x) = kx \), where \( k \) is called the spring constant or the stiffness constant. To calculate the work required to stretch the spring a length \( x \) we use
\[
W = \int_a^b F(x)dx.
\]
where \( a \) is the initial displacement of the spring (\( a = 0 \) if the spring is initially unstretched) and \( b \) is the final displacement. A force of 5 N is exerted on a spring and stretches it 1 m beyond its natural length.
   a. Find the spring constant \( k \).
   b. How much work is required to stretch the spring 1.8 m beyond its natural length?

5. When a force of 30 N is applied to a spring, it stretches it from a length of 12 cm to 15 cm. How much work will be done in stretching the spring from 12 cm to 20 cm? (Hint: read the first part of problem #4 above.)
6. A flat surface is submerged vertically in a fluid of weight density \( w \). If the weight density \( w \) is doubled, is the force on the plate also doubled? Explain.

7. The bottom of a rectangular swimming pool, whose bottom is an inclined plane, is shown below. Calculate the fluid force on the bottom of the pool when it is filled completely with water.

![Diagram of a swimming pool](image)

8. Suppose \( f(x) \) is the probability density function for the lifetime of a manufacturer’s light bulb, where \( x \) is measured in hours. Explain the meaning of each integral.
   a. \( \int_{1000}^{5000} f(x) \, dx \)
   b. \( \int_{3000}^{5000} f(x) \, dx \)

9. The length of time a customer spends waiting until his/her entrée is served at a certain restaurant is modeled by an exponential density function with an average time of 8 minutes.
   a. What is the probability that a customer is served in the first 3 minutes?
   b. What is the probability that a customer has to wait more than 10 minutes?

10. The average height of an adult female in Los Angeles is 63.4 inches (5 feet 3.4 inches) with a standard deviation of 3.2 inches.
    a. What is the probability that a female’s height is less than 63.4 inches?
    b. What is the probability that a female’s height is between 63 and 65 inches?
    c. What is the probability that a female’s height is more than 6 feet?
    d. What is the probability that a female’s height is exactly 5 feet?

**Review Answers**

1. 1.471 J
2. 0 J
3. \( 5 \times 10^{10} \) J
4. 
   a. \( k = 5 \) N/m
   b. 8.1 J
5. 3.2 J
6. Yes. To explain why, ask how \( w \) and \( F \) are mathematically related.
7. 63,648 N
8. 
   a. The probability that a randomly chosen light bulb will have a lifetime between 1000 and 5000 hours.
   b. The probability that a randomly chosen light bulb will have a lifetime of at least 3000 hours.
9. 
   a. 31%
   b. 29%
10. 
    a. 50%
    b. 24%
    c. 0.36%
    d. almost 0%

**Texas Instruments Resources**

*In the CK-12 Texas Instruments Calculus FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See [http://www.ck12.org/flexr/chapter/9730](http://www.ck12.org/flexr/chapter/9730).*